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The U(16) algebraic lattice: II. analytic construction

Dimitri Kusnezov

NSCL and Department of Physics, Michigan State University, East Lansing, MI 48824-1321, USA

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Abstract. We construct the dynamical symmetries of U(16), the algebra related to the quantization of classical reflection asymmetric shapes constructed from $\lambda^{P}=0^{+}, 1^{-}, 2^{+}, 3^{-}$ multipoles. Generators, Casimir operators and their expectation values and branching rules are detailed for all dynamical symmetry limits, focusing primarily on those relevant to octupole deformations and vibrations. The resulting operators allow the construction of dynamical symmetry Hamiltonians.

1. Introduction

The U(16) algebra was originally proposed to describe the bosonic quantization of classical reflection asymmetric shapes described by $\lambda^{P}=0^{+}$, 1^{-}, 2^{+} and 3^{-} multipoles [1]. The application was directed to the description of nuclei with reflection-asymmetric ground states. Recently, this algebra and its non-compact extension U(15, 1) have also been proposed to describe bag-like properties of baryonic spectra [2]. Such interacting boson algebraic approaches to collective excitations have had tremendous success in describing many features of nuclei [3]. To date, nuclear interacting boson model studies have focused almost exclusively on positive-parity excitations ($\lambda^{P}=0^{+}$, 2^{+}). This is due in part to the lack of a model to describe the interactions with negative parity excitations and the correct electromagnetic transition properties observed in nuclei. The current interest in nuclei with stable ground-state octupole deformation has resulted in a flourish of experiments in the light actinides and neutron-rich rare-earths, providing new data on the interaction between positive- and negative-parity collective nuclear excitations. This has led to the emergence of interacting-boson-model Hamiltonians constructed from positive- (s and d) and negative- (p and f) parity bosons [4, 5], which has had some success in describing experimental data in nuclei with suspected octupole deformation [5-7]. The success of this model indicates the need for a full understanding of the algebraic properties and dynamical symmetries of U(16), which can result in tractable analysis and predictions of experimental data.

In a previous article [8] (hereafter referred to as I), the complete algebraic lattice of U(16) was constructed. There we argued that only seven of the 165 dynamical symmetry limits of U(16), called pdf dynamical symmetry limits, are relevant to the study of octupole deformed shapes. Unfortunately the identification of the subalgebra lattice in I is only the first step in the study of predictions of the U(16) model. The explicit structure of the subalgebra, specifically the forms of the generators, the quadratic Casimir invariants and the branching rules, are of central importance to the study of dynamical symmetry predictions. In this article we complete the next step of the analysis of U(16): the explicit construction of the generators and quadratic Casimir invariants of all the subalgebras. We focus primarily on the pdf dynamical symmetry limits, although for completeness all algebras are constructed. With the completion of this step, the final step will be to detail the physical structure of the relevant symmetries using simple phenomenological Hamiltonians built from the Casimir operators. Electromagnetic transition operators can then be selected from the generators of the dynamical symmetry algebras of the particular chain. This will be the subject of a separate study.

We begin in section 2 with the introduction of the U(16) interacting boson model. In section 3 the maximal subalgebras of U(16) are constructed. In section 4 the seven pdf dynamical symmetry limits, introduced in I, are studied. The remaining non-trivial subalgebras that appear in the dynamical symmetry limits of U(16) are discussed briefly in section 5, followed by the conclusions in section 6. As is now standard in interacting boson model literature, we refer only to the classical Lie algebras with the large symbols U(N), O(N), Sp(N) and G_2.

2. The spdf IBM and its generators

The spdf IBM is generated by boson creation and annihilation operators with $J^\pi = 0^+$, 1^-, 2^+, 3^-. Their creation (annihilation) operators are denoted by $s^\dagger$ (s), $p^\dagger_\mu$ (p_μ), $d^\dagger_\mu$ (d_μ) and $f^\dagger_\mu$ (f_μ), respectively, where μ is the magnetic projection ($\mu = -l, \ldots, l$). Occasionally, we use the generic operators $b_{l,\mu}$ and $b^\dagger_{l,\mu}$ with $l = 0, 1, 2, 3$. It is well known that while the creation operators transform as spherical tensors, the annihilation operators do not. This leads to a redefinition of annihilation operators that do transform as spherical tensors: $\tilde{b}_{l,\mu} = (-1)^{l+\mu} b_{l,-\mu}$. The generators of the Lie algebra are then constructed from the Racah tensors:

$$G^{(K)}_Q(ll') \equiv [b^\dagger_l \tilde{b}_{l'}]^{(K)}_Q = \sum_{\mu\mu'} \langle l\mu l'\mu' | KQ\rangle b^\dagger_{l\mu} \tilde{b}_{l'\mu'}. \tag{1}$$

Occasionally it is convenient to construct subalgebras from pseudo-spin realizations of U(16), using double tensors of the form [6]

$$\hat{\mathscr{G}}^{(K_1K_2)}_{Q_1Q_2} = \sum_{L_1,L_2,\alpha,\beta} P(L_1, L_2)\langle K_1Q_1K_2Q_2|\alpha\beta\rangle \begin{Bmatrix} j_1 & j_2 & L_1 \\ j_3 & j_4 & L_2 \\ K_1 & K_2 & \alpha \end{Bmatrix}$$
$$\times \sqrt{(2K_1+1)(2K_2+1)(2K_3+1)(2K_4+1)}\, G^\alpha_\beta(L_1L_2) \tag{2}$$

where $P(L_1, L_2)$ is an arbitrary phase. This type of realization of U(16) is also relevant in the study of Bose–Fermi symmetries in odd-odd nuclei. Particularly, U(16) arises as the symmetry group when two quasi-particles corresponding to neutron and proton degrees of freedom are given $j = \frac{3}{2}$ configurations [9]. Upon constructing generators that mix the protons and neutrons, the U(16) algebra is obtained as a maximal symmetry. The subalgebra U(4)⊕U(4) then emerges as a natural decomposition of U(16) into separate proton and neutron U(4) subalgebras, one for each $j = \frac{3}{2}$ configuration. The similarity in the algebraic structure is related to the fact that bosons with $\lambda = 0$, 1, 2 and 3 can be expressed as coupled fermions, each with $j = \frac{3}{2}$. In this respect, the algebraic lattice of U(16) that is detailed in this article is relevant to such Bose–Fermi models. The Racah tensors satisfy the commutation relations

$$
\begin{aligned}
&[G_Q^{(K)}(l_1l_2), G_{Q'}^{(K')}(l_3l_4)] \\
&\quad = \sum_{K'',Q''} \sqrt{(2K+1)(2K'+1)}\langle KQK'Q'|K''Q''\rangle \\
&\quad \times(-1)^{K'-K}\left[\delta_{l_2l_3}(-1)^{l_1-l_4+K+K'+K''}\begin{Bmatrix} K & K' & K'' \\ l_4 & l_1 & l_2 \end{Bmatrix} G_{Q''}^{(K'')}(l_1l_4)\right. \\
&\quad \left. -\delta_{l_1l_4}(-1)^{l_3-l_2}\begin{Bmatrix} K & K' & K'' \\ l_3 & l_2 & l_1 \end{Bmatrix} G_{Q''}^{(K'')}(l_3l_2)\right].
\end{aligned}
\tag{3}
$$

Here the symbol with the curly brackets is the usual 6-J symbol found in [10]. The physical angular momentum algebra $O_{pdf}(3)$ is generated by

$$\hat{L}_M^{(1)} = \sqrt{2}[p^\dagger\tilde{p}]_M^{(1)} + \sqrt{10}[d^\dagger\tilde{d}]_M^{(1)} + 2\sqrt{7}[f^\dagger\tilde{f}]_M^{(1)} \tag{4}$$

and must appear in every subalgebra chain. The general chains are of the form $U_{spdf}(16) \supset G \supset G' \ldots \supset O_{pdf}(3)$. Here G and G$'$ are intermediate subalgebras. The O(2) subalgebra is not important in our constructions, and since it can be added trivially to every chain, we omit it.

3. Maximal subalgebras of $U_{spdf}(16)$

In this section we construct the maximal subalgebras of $U_{spdf}(16)$, listed in table 1, and branching rules. The simplest subalgebras are the maximal regular subalgebras, which correspond to partitions of the four types of bosons into separate subalgebras [8, 11, 12]. The less trivial S-subalgebras require pseudospin realizations of $U_{spdf}(16)$ that better reflect the dynamical symmetry. In this section we refer to the generators listed in table 2.

Table 1. Maximal simple and non-simple regular and S-subalgebras of the $U_{spdf}(16)$ interacting boson model which contain the physical angular momentum.

Model	Dynamical symmetry	Type of embedding
$U_{spdf}(16)$		S-subalgebras:
	$O_{spdf}(16)$	(Simple)
	$O_{spdf}(10)$	(Spinor, simple)
	$SU_{spdf}(4)\oplus SU_{spdf}(4)$	(Non-simple)
	$SU_{spdf}(2)\oplus SU_{spdf}(8)$	(Non-simple)
		Regular subalgebras:
	$U_{pdf}(15)$	(Simple)
	$U_p(3)\oplus U_{sdf}(13)$	(Non-simple)
	$U_{sp}(4)\oplus U_{df}(12)$	(Non-simple)
	$U_d(5)\oplus U_{spf}(11)$	(Non-simple)
	$U_{sd}(6)\oplus U_{pf}(10)$	(Non-simple)
	$U_f(7)\oplus U_{spd}(9)$	(Non-simple)
	$U_{sf}(8)\oplus U_{pd}(8)$	(Non-simple)

Table 2. Generators and quadratic invariants of selected maximal subalgebras of $U_{spdf}(16)$.

SU(4)⊕SU(4)	$\hat{L}^{(1)}_\mu=\sqrt{20}\mathscr{H}^{(1)}_{\mu+}=\sqrt{2}[p^\dagger\tilde{p}]^{(1)}_\mu+\sqrt{10}[d^\dagger\tilde{d}]^{(1)}_\mu+2\sqrt{7}[f^\dagger\tilde{f}]^{(1)}_\mu$
	$\hat{D}^{(1)}_\mu=-\sqrt{20}\mathrm{i}\mathscr{H}^{(1)}_{\mu-}=\sqrt{5}[s^\dagger\tilde{p}+p^\dagger s]^{(1)}_\mu-2\sqrt{2}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]^{(1)}+\sqrt{7}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]^{(1)}_\mu$
	$\hat{Q}^{(2)}_\mu=\mathscr{H}^{(2)}_{\mu+}=\frac{1}{2}[s^\dagger\tilde{d}+d^\dagger\tilde{s}]^{(2)}_\mu+(\sqrt{21}/10)[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(2)}_\mu-(\sqrt{6}/5)[p^\dagger\tilde{p}]^{(2)}_\mu+(\sqrt{21}/5)[f^\dagger\tilde{f}]^{(2)}_\mu$
	$\hat{M}^{(2)}_\mu=-\mathrm{i}\mathscr{H}^{(2)}_{\mu-}=-\sqrt{3/10}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]^{(2)}_\mu+\sqrt{7/10}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]^{(2)}_\mu$
	$\hat{T}^{(3)}_\mu=\mathscr{H}^{(3)}_{\mu+}=\sqrt{3/5}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(3)}_\mu-(1/\sqrt{2})[d^\dagger\tilde{d}]^{(3)}_\mu+\sqrt{3/10}[f^\dagger\tilde{f}]^{(3)}_\mu$
	$\hat{O}^{(3)}_\mu=-\mathrm{i}\mathscr{H}^{(3)}_{\mu-}=\frac{1}{2}[s^\dagger\tilde{f}+f^\dagger\tilde{s}]^{(3)}_\mu+\frac{1}{2}\sqrt{3/5}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]^{(3)}_\mu+\sqrt{3/5}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]^{(3)}_\mu$
	$\hat{C}_{2,SU_a(4)\oplus SU_b(4)}=\frac{1}{4}(\frac{1}{20}\hat{L}^2+\hat{Q}^2+\hat{T}^2-\frac{1}{20}\hat{D}^2-\hat{M}^2-\hat{O}^2)$
O(10)	$\hat{\mathscr{D}}^{(0)}_0=(1/4\sqrt{5})(5\hat{n}_s+\hat{n}_p-3\hat{n}_d+\hat{n}_f)$
	$\hat{\mathscr{D}}^{(1)}_\mu=(1/2\sqrt{10})(\sqrt{5}[s^\dagger\tilde{p}-p^\dagger s]^{(1)}_\mu-2\sqrt{2}[p^\dagger\tilde{d}-d^\dagger\tilde{p}]^{(1)}_\mu-\sqrt{7}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(1)}_\mu)$
	$\hat{\mathscr{D}}^{(2)}_\mu=\frac{1}{10}(\sqrt{21}[p^\dagger\tilde{p}]^{(2)}_\mu+5[d^\dagger\tilde{d}]^{(2)}_\mu-\sqrt{6}[f^\dagger\tilde{f}]^{(2)}_\mu+2\sqrt{6}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(2)}_\mu)$
	$\hat{\mathscr{D}}^{(3)}_\mu=-(1/2\sqrt{2})([s^\dagger\tilde{f}-f^\dagger s]^{(3)}_\mu+\sqrt{3/5}[p^\dagger\tilde{d}-d^\dagger\tilde{p}]^{(3)}_\mu-2\sqrt{3/5}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(3)}_\mu)$
	$\hat{\mathscr{D}}^{(4)}_\mu=\frac{1}{2}([d^\dagger\tilde{d}]^{(4)}_\mu-\sqrt{2/5}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(4)}_\mu+\sqrt{11/5}[f^\dagger\tilde{f}]^{(4)}_\mu)$
	$\hat{L}^{(1)}_\mu=\sqrt{2}[p^\dagger\tilde{p}]^{(1)}_\mu+\sqrt{10}[d^\dagger\tilde{d}]^{(1)}_\mu+2\sqrt{7}[f^\dagger\tilde{f}]^{(1)}_\mu$
	$\hat{D}^{(1)}_\mu=\sqrt{5}[s^\dagger\tilde{p}+p^\dagger s]^{(1)}_\mu-2\sqrt{2}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]^{(1)}+\sqrt{7}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]^{(1)}_\mu$
	$\hat{T}^{(3)}_\mu=\sqrt{3/5}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(3)}_\mu-(1/\sqrt{2})[d^\dagger\tilde{d}]^{(3)}_\mu+\sqrt{3/10}[f^\dagger\tilde{f}]^{(3)}_\mu$
	$\hat{O}^{(3)}_\mu=\frac{1}{2}[s^\dagger\tilde{f}+f^\dagger\tilde{s}]^{(3)}_\mu+\frac{1}{2}\sqrt{3/5}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]^{(3)}_\mu+\sqrt{3/5}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]^{(3)}_\mu$
	$\hat{C}_{2,O(10)}=\frac{1}{4}\left(\hat{\mathscr{G}}^{(10)}\cdot\hat{\mathscr{G}}^{(10)}+\hat{\mathscr{G}}^{(30)}\cdot\hat{\mathscr{G}}^{(30)}+\hat{\mathscr{G}}^{(01)}\cdot\hat{\mathscr{G}}^{(01)}+\hat{\mathscr{G}}^{(03)}\cdot\hat{\mathscr{G}}^{(03)}+\sum_{\mu\nu}(-1)^{\mu+\nu}\hat{\mathscr{G}}^{(22)}_{\mu\nu}\hat{\mathscr{G}}^{(22)}_{-\mu,-\nu}\right)$
	$=\frac{1}{4}\left(\frac{1}{20}\hat{L}^2+\hat{T}^2-\frac{1}{20}\hat{D}^2-\hat{O}^2+\sum_{K=0}^{4}\hat{\mathscr{D}}^{(K)}\cdot\hat{\mathscr{D}}^{(K)}\right)$
SU(8)⊕*SU*(2)	$\hat{\mathscr{H}}^{(10)}_{M0}=\frac{1}{3}\sqrt{7/2}[d^\dagger\tilde{d}]^{(1)}_M+(\sqrt{5}/3)[f^\dagger\tilde{f}]^{(1)}_M-\frac{1}{6}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(1)}_M$
	$\hat{\mathscr{H}}^{(20)}_{M0}=(1/\sqrt{3})[d^\dagger\tilde{d}]^{(2)}_M+(1/\sqrt{2})[f^\dagger\tilde{f}]^{(2)}_M-(1/2\sqrt{3})[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(2)}_M$
	$\hat{\mathscr{H}}^{(30)}_{M0}=\frac{1}{2}[d^\dagger\tilde{d}]^{(3)}_M+\frac{1}{2}\sqrt{5/3}[f^\dagger\tilde{f}]^{(3)}_M-(1/\sqrt{6})[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(3)}_M$
	$\hat{\mathscr{H}}^{(40)}_{M0}=(\sqrt{5}/6)[d^\dagger\tilde{d}]^{(4)}_M+(\sqrt{11}/6)[f^\dagger\tilde{f}]^{(4)}_M-\frac{1}{3}\sqrt{5/2}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(4)}_M$
SU(8)	$\hat{\mathscr{H}}^{(50)}_{M0}=(1/\sqrt{6})[f^\dagger\tilde{f}]^{(5)}_M-\frac{1}{2}\sqrt{5/3}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(5)}_M$
	$\hat{\mathscr{I}}^{(30)}_{M0}\pm\hat{\mathscr{J}}^{(30)}_{M0}=-\frac{1}{2}[s^\dagger\tilde{f}\mp f^\dagger s]^{(3)}_M+\frac{1}{2}\sqrt{5/3}\,[p^\dagger\tilde{d}\mp d^\dagger\tilde{p}]^{(3)}_M+(1/\sqrt{3})[p^\dagger\tilde{f}\pm f^\dagger\tilde{p}]^{(3)}_M$
	$\hat{\mathscr{I}}^{(20)}_{M0}\pm\hat{\mathscr{J}}^{(20)}_{M0}=\frac{1}{2}[s^\dagger\tilde{d}\pm d^\dagger s]^{(2)}_M+(1/\sqrt{6})[p^\dagger\tilde{d}\mp d^\dagger\tilde{p}]^{(2)}_M+\frac{1}{2}\sqrt{7/3}[p^\dagger\tilde{f}\pm f^\dagger\tilde{p}]^{(2)}_M$
	$\sqrt{3}\hat{\mathscr{L}}^{(00)}_{00}-\hat{\mathscr{H}}^{(00)}_{00}=(1/2\sqrt{3})(3\hat{n}_s+3\hat{n}_p-\hat{n}_d-\hat{n}_f)$
	$\hat{\mathscr{L}}^{(10)}_{M0}=(1/\sqrt{2})[p^\dagger\tilde{p}]^{(1)}_M-\frac{1}{2}[s^\dagger\tilde{p}-p^\dagger s]^{(1)}_M$
	$\hat{C}_{2,SU(8)}=\frac{1}{8}\left(\sum_{K=1}^{5}\hat{\mathscr{H}}^{(K0)}\cdot\hat{\mathscr{H}}^{(K0)}+\hat{\mathscr{L}}^{(10)}\cdot\hat{\mathscr{L}}^{(10)}+\sum_{K=2,3}(\hat{\mathscr{I}}^{(K0)}\cdot\hat{\mathscr{J}}^{(K0)}+\hat{\mathscr{J}}^{(K0)}\cdot\hat{\mathscr{I}}^{(K0)})\right)$
	$+\frac{1}{32}(\sqrt{3}\hat{\mathscr{L}}^{(00)}_{00}-\hat{\mathscr{H}}^{(00)}_{00})\cdot(\sqrt{3}\hat{\mathscr{L}}^{(00)}_{00}-\hat{\mathscr{H}}^{(00)}_{00})$
SU(2)	$\sqrt{3}\hat{\mathscr{H}}^{(01)}_{0Q}+\hat{\mathscr{L}}^{(01)}_{0Q}=(\sqrt{7}/3)[f^\dagger\tilde{f}]^{(1)}_M-\frac{1}{3}\sqrt{5/2}[d^\dagger\tilde{d}]^{(1)}_M+\frac{1}{6}\sqrt{35}[d^\dagger\tilde{f}-f^\dagger\tilde{d}]^{(1)}_M$
	$+(1/\sqrt{2})[p^\dagger\tilde{p}]^{(1)}_M+\frac{1}{2}[s^\dagger\tilde{p}-p^\dagger s]^{(1)}_M$
	$\hat{C}_{2,SU(2)}=\frac{1}{4}(\sqrt{3}\hat{\mathscr{H}}^{(01)}+\hat{\mathscr{L}}^{(01)})\cdot(\sqrt{3}\hat{\mathscr{H}}^{(01)}+\hat{\mathscr{L}}^{(01)})$

3.1. Regular subalgebras

The simple and non-simple regular subalgebras are formed from different groupings of the four types of bosons. There are seven such maximal subalgebras: $U_{pdf}(15)$, $U_{sdf}(13)\oplus U_p(3)$, $U_{df}(12)\oplus U_{sp}(4)$, $U_{spf}(11)\oplus U_d(5)$, $U_{pf}(10)\oplus U_{sd}(6)$, $U_{spd}(9)\oplus U_f(7)$, $U_{sf}(8)\oplus U_{pd}(8)$. The generators of each of these algebras are all combinations of operators $[b_j^\dagger\tilde{b}_k]^{(L)}_M$, where j and k range over the bosons indicated in the subscripts of the algebras. The Casimir operators and the branching rules for these algebras are summarized in the appendix.

3.2. *The $O_{spdf}(16)$ subalgebra*

The generators and quadratic Casimir invariants of $O_{spdf}(16)$ are given by the constructions in the appendix. Choosing the phase convention (A.3) (used throughout the article), the 120 generators are: $[f^\dagger s+s^\dagger\tilde{f}]_M^{(3)}$, $[p^\dagger\tilde{p}]_M^{(1)}$, $[d^\dagger\tilde{d}]_M^{(1,3)}$, $[f^\dagger\tilde{f}]_M^{(1,3,5)}$, $[p^\dagger s+s^\dagger\tilde{p}]_M^{(1)}$, $[d^\dagger s+s^\dagger\tilde{d}]_M^{(2)}$, $[d^\dagger\tilde{p}+(-1)^L p^\dagger\tilde{d}]_M^{(1,2,3)}$, $[f^\dagger\tilde{p}-(-1)^L p^\dagger\tilde{f}]_M^{(2,3,4)}$ and $[f^\dagger\tilde{d}+(-1)^L d^\dagger\tilde{f}]_M^{(1,\ldots,6)}$. The quadratic Casimir operator has the form

$$\hat{C}_{2,\mathrm{O}(16)}=\tfrac{1}{28}[\hat{N}_{\mathrm{tot}}(\hat{N}_{\mathrm{tot}}+14)-\hat{P}^\dagger\cdot\hat{P}] \tag{5}$$

where $\hat{P}^\dagger=-s^\dagger\cdot s^\dagger-p^\dagger\cdot p^\dagger+d^\dagger\cdot d^\dagger-f^\dagger\cdot f^\dagger$. The expectation value in the fully symmetric representation of $O_{spdf}(16)$ is

$$\langle\hat{C}_{2,\mathrm{O}(16)}\rangle=\tfrac{1}{28}v_{\mathrm{spdf}}(v_{\mathrm{spdf}}+14) \tag{6}$$

and the restriction $U_{spdf}(16)\supset O_{spdf}(16)$ on the Dynkin labels are $[N_{\mathrm{tot}}]\to(v_{\mathrm{spdf}},0,0,0,0,0,0,0)$, where $v_{\mathrm{spdf}}=0,1,\ldots,N_{\mathrm{tot}}$.

3.3. *The SU(4)⊕SU(4) subalgebra*

It is more convenient to introduce a pseudo-spin construction of U(16) that better reflects the symmetry of this decomposition, using $j_i=\frac{3}{2}$ $(i=1,\ldots,4)$, and $P(L_1,L_2)=(-1)^{(1/2)(L_1^2+L_2^2)+L_2}$ in (2). The generators $\hat{\mathscr{G}}_{Q_10}^{(K_10)}$ and $\hat{\mathscr{G}}_{0Q_2}^{(0K_2)}$ provide the realization of SU(4)⊕SU(4). Generators of good rank and parity can be defined by $\mathscr{H}_{Q,\pm}^{(K)}=\hat{\mathscr{G}}_{Q0}^{(K0)}\pm\hat{\mathscr{G}}_{0Q}^{(0K)}$. The rank-zero operator is omitted since it is the same for both SU(4) subalgebras: $\hat{\mathscr{G}}_{00}^{(00)}=\hat{N}_{\mathrm{tot}}/4$. (Of course it could be added to one of the algebras to produce SU(4)⊕U(4).) The quadratic Casimir operators of each SU(4) subalgebra are of mixed parity and cannot appear alone in the Hamiltonian, which must be scalar, rotationally invariant and Hermitian. A suitable operator can be obtained by summing the two operators. The Young label branching rules for the decomposition $U_{spdf}(16)\supset$ SU(4)⊕SU(4) are given by $[N_{\mathrm{tot}}]\to[n_1,n_2,n_3]\otimes[n_1,n_2,n_3]$, were $n_1+n_2+n_3=N_{\mathrm{tot}}-4\kappa$, $n_1\geqslant n_2\geqslant n_3\geqslant 0$ and $\kappa=0,1,\ldots,[N_{\mathrm{tot}}/4]$. Here the symbol $[N_{\mathrm{tot}}/4]$ denotes the largest integer less than or equal to the ratio. The expectation value of the operator is just the sum of two SU(4) expectation values in the above representations:

$$\langle\hat{C}_{2,\mathrm{SU}(4)\oplus\mathrm{SU}(4)}\rangle=\tfrac{1}{16}[3(n_1^2+n_2^2+n_3^2)-2(n_1n_2+n_2n_3+n_3n_1)+4(3n_1+n_2-n_3)]. \tag{7}$$

3.4. *The O(10) spinor subalgebra*

This dynamical symmetry limit corresponds to the embedding of the spinor representation (00001) of O(10) into the fully symmetric representation [1] of $U_{spdf}(16)$. The 45 generators of O(10) are $\hat{\mathscr{G}}_{\mu\nu}^{(22)}$, $\hat{\mathscr{G}}_{0\mu}^{(01)}$, $\hat{\mathscr{G}}_{\mu0}^{(10)}$, $\hat{\mathscr{G}}_{\mu0}^{(30)}$ and $\hat{\mathscr{G}}_{0\mu}^{(03)}$, using the realization in subsection 3.3. This algebra is a result of the structural zero of the 6-j coefficient

$$\begin{Bmatrix}2 & 2 & 2\\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2}\end{Bmatrix}=0. \tag{8}$$

The 25 generators of $\hat{\mathscr{G}}^{(22)}_{\mu\nu}$ are of mixed parity and rank, and a better representation of O(10) can be obtained from linear combinations of the $\hat{\mathscr{G}}^{(22)}_{\mu\nu}$, by inverting the definition of $\hat{\mathscr{G}}^{(22)}_{\mu\nu}$:

$$\hat{\mathscr{G}}^{(22)}_{\mu\nu} = \sum_{K,Q} \langle 2\mu 2\nu | KQ\rangle (-1)^{-K^2/2} \mathscr{D}^{(K)}_Q . \tag{9}$$

The quadratic Casimir expectation value is given by

$$\langle \hat{C}_{2,\mathrm{O}(10)}\rangle = \tfrac{1}{16}[5v_{10}^2 - 4v_{10}(N_{\mathrm{tot}}-2) + N_{\mathrm{tot}}^2] \tag{10}$$

where Dynkin label branching rules for fully symmetric representations of U(16) into O(10) are given by $[N_{\mathrm{tot}}] \to (v_{10}, 0, 0, 0, N_{\mathrm{tot}} - 2v_{10})$, with $v_{10} = 0, 1, \ldots, [N_{\mathrm{tot}}/2]$.

3.5. The SU(8)⊕SU(2) subalgebra

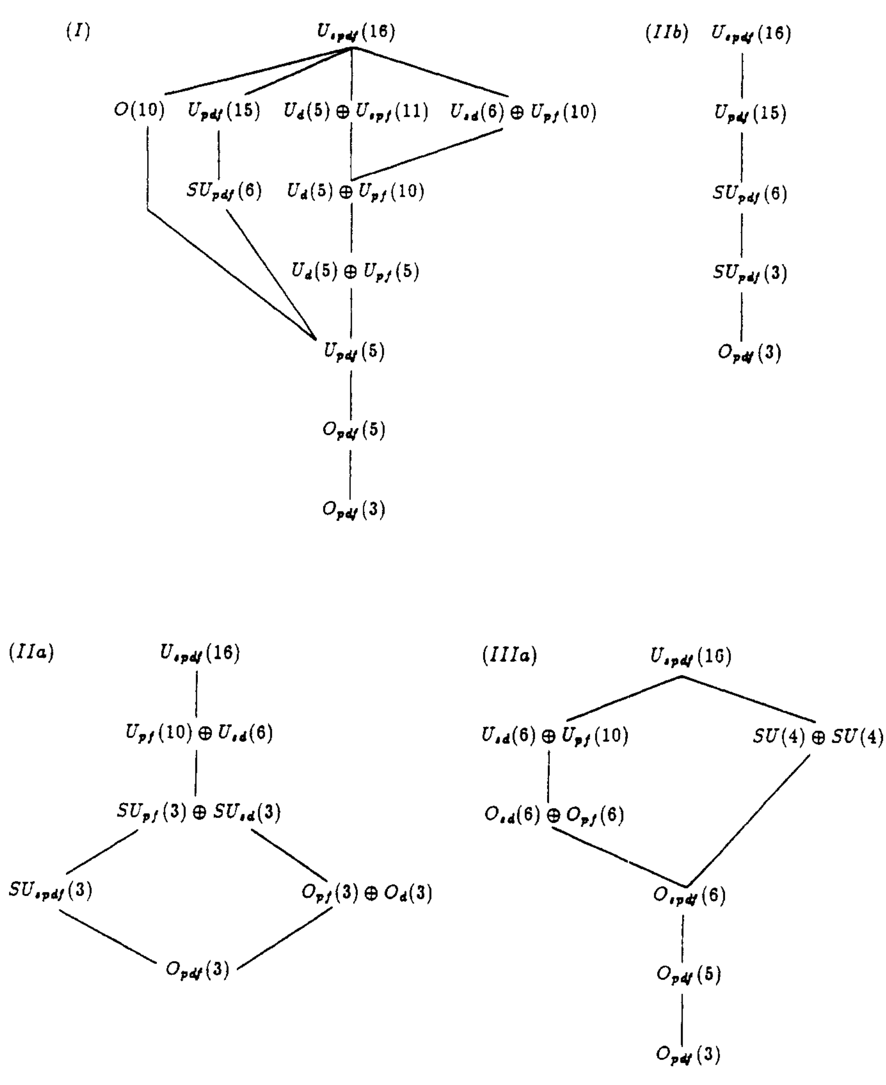
This decomposition is easily obtained through the pseudo-spin generators of (2), where $j_i = \frac{1}{2}, \frac{5}{2}$. Specifically we denote the operators as $\hat{\mathscr{H}}^{(KK')}_{QQ'}$ $(j_1 = j_3 = \frac{5}{2}, j_2 = j_4 = \frac{1}{2})$, $\hat{\mathscr{I}}^{(KK')}_{QQ'}$ $(j_1 = j_2 = j_4 = \frac{1}{2}, j_3 = \frac{5}{2})$, $\hat{\mathscr{J}}^{(KK')}_{QQ'}$ $(j_2 = j_3 = j_4 = \frac{1}{2}, j_1 = \frac{5}{2})$ and $\hat{\mathscr{L}}^{(KK')}_{QQ'}$ $(j_1 = j_2 = j_3 = j_4 = \frac{1}{2})$. Of these 256 generators, SU(8) is generated by the 63 generators with only left indices while SU(2) is generated by three generators with right indices. The branching rules for $\mathrm{U_{spdf}}(16) \supset \mathrm{SU}(8) \oplus \mathrm{SU}(2)$ in terms of Young labels for SU(8) and the spin for SU(2) are given by $[N_{\mathrm{tot}}] \to [N_{\mathrm{tot}}/2 + \tilde{j}, N_{\mathrm{tot}}/2 - \tilde{j}, 0, 0, 0, 0, 0] \otimes [\tilde{j}]$, where $\tilde{j} = N_{\mathrm{tot}}/2$, $N_{\mathrm{tot}}/2 - 1, \ldots, \frac{1}{2}$ or 0. The quadratic Casimir operators have the expectation values

$$\begin{aligned} \langle \hat{C}_{2,\mathrm{SU}(8)}\rangle &= \tfrac{1}{128}[3N_{\mathrm{tot}}(N_{\mathrm{tot}}+16) + 16\tilde{j}(\tilde{j}+1)] \\ \langle \hat{C}_{2,\mathrm{SU}(2)}\rangle &= \tfrac{1}{2}\tilde{j}(\tilde{j}+1). \end{aligned} \tag{11}$$

4. pdf dynamical symmetry limits

In this section we systematically detail the pdf dynamical symmetry limits [8]. These dynamical symmetries are the limits of the $U_{\mathrm{spdf}}(16)$ model which do not decouple p and f bosons, and retain interactions between positive- and negative-parity bosons for subalgebras with rank greater than 1. Octupole deformed systems can only be formed from generators that mix s-d and p-f bosons, or equivalently by dynamical symmetry limits in which the generators $\hat{\mathscr{G}}$ do not conserve the number of negative-parity bosons: $[\hat{\mathscr{G}}, \hat{N}_-] \neq 0$. We can consider the three classes of algebras introduced in I. Class A is defined as the generators (and hence Casimir invariants) of the algebras that appear as subalgebras of $U_{\mathrm{sd}}(6) \oplus U_{\mathrm{pf}}(10)$ and necessarily separately conserve the number of negative- and positive-parity bosons. Hamiltonians constructed from the invariants of these subalgebras generate states of well defined parity, given by the expectation value of the number operator for negative-parity bosons, $\langle \hat{N}_- \rangle$. Nearly all algebras in table 3, including the entire $\mathrm{SU_{spdf}}(3)$ limit (IIa), fall into this class. Naturally parity doublets are not manifest in these limits since negative-parity states can be moved with respect to positive-parity states with the linear invariants of $\mathrm{U_{sd}}(6)$ and $\mathrm{U_{pf}}(10)$. The remaining algebras are the exceptions to this class. Class B are the algebras with Casimir operators that have good parity but do not commute with $\hat{N}_-$, while class C are the algebras with Casimir operators that are of mixed parity and hence do not commute with $\hat{N}_-$. There are only two algebras in table 3 of class C: $\mathrm{SU_{pdf}}(6)$ and $\mathrm{SU_{pdf}}(3)$. The eight

Table 3. The pdf dynamical symmetry limits of $U_{spdf}(16)$.



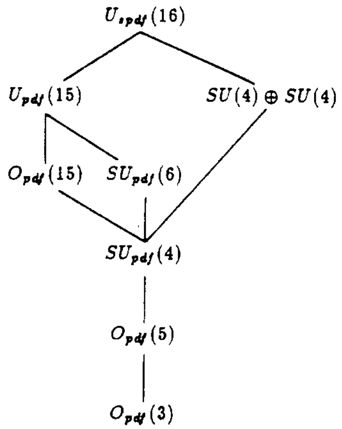
class B algebras are $SU(4)\oplus SU(4)$, $Sp(4)\oplus Sp(4)$, $O_{spdf}(16)$, $O_{pdf}(15)$, $O_{spf}(11)$, $O(10)$, $O_{spdf}(4)$ and $SU_{pdf}(4)$. From this simple classification, it is clear that Hamiltonians that describe octupole deformation must include terms from class B or C. Only in this way can negative-parity bosons be mixed into the ground-state wavefunction. The generators are all summarized in tables 3–9.

4.1. The $SU_{pdf}(5)$ limit (I)

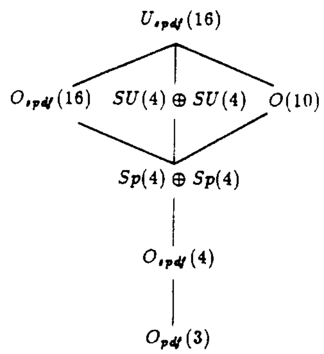
Aside from the $O(10)$ and $SU_{pdf}(6)$ subalgebras which are of classes B and C, all the subalgebras are of class A, and hence conserve separately the number of pf and sd

Table 3. (continued)

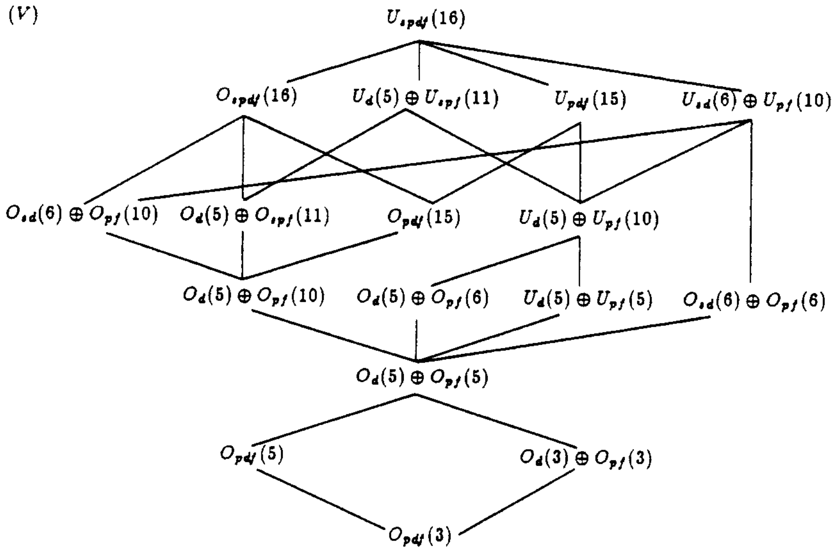
(*IIIb*)



(*IV*)



(*V*)



bosons. This limit contains quadrupole vibrations coupled to the octupole/dipole algebra $U_{pf}(5)$. Octupole deformation can only be described in this system by Hamiltonians that include operators from O(10) or $SU_{pdf}(6)$. The four maximal subalgebras O(10), $U_{pdf}(15)$, $U_d(5) \oplus U_{spf}(11)$ and $U_{sd}(6) \oplus U_{pf}(10)$, as well as $U_d(5) \oplus U_{pf}(10)$ have already been detailed. The remaining subalgebra at this level is $SU_{pdf}(6)$, for which the generators are listed in table 4. Unlike the number operator, the scalar operator $\hat{N}_0^{(0)}$ does not commute with $\hat{F}^{(2)}$ or $\hat{M}^{(2)}$. (The phases of $\hat{\mathscr{D}}^{(2)}$ and $\hat{\mathscr{D}}^{(4)}$ are slightly different from those used in the O(10) construction.) The embedding $SU_{pdf}(15) \supset SU_{pdf}(6)$ for fully symmetric representations of SU(15) in Dynkin labels [13] is $[N_{pdf}] \rightarrow [0, n_1, 0, N_{pdf} - 2n_1 - 3n_2, 0]$, where the sum over all non-negative integers n_1 and n_2 is

Table 4. Generators and quadratic invariants of the $U_{pdf}(5)$ limit (I). Refer to table 3 for subalgebra embeddings.

$U_{pdf}(6)$	$\hat{N}_0^{(0)} = 2\hat{n}_d - \hat{n}_p - \hat{n}_f$ $\hat{L}_\mu^{(1)} = \sqrt{2}[p^\dagger\tilde{p}]_\mu^{(1)} + \sqrt{10}[d^\dagger\tilde{d}]_\mu^{(1)} + 2\sqrt{7}[f^\dagger\tilde{f}]_\mu^{(1)}$ $\hat{\mathscr{D}}_\mu^{(2)} = \frac{1}{10}(\sqrt{21}[p^\dagger\tilde{p}]_\mu^{(2)} - 5[d^\dagger\tilde{d}]_\mu^{(2)} - \sqrt{6}[f^\dagger\tilde{f}]_\mu^{(2)} + 2\sqrt{6}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(2)})$ $\hat{M}_\mu^{(2)} = -\sqrt{3/10}[p^\dagger\tilde{d} + d^\dagger\tilde{p}]_\mu^{(2)} + \sqrt{7/10}[d^\dagger\tilde{f} + f^\dagger\tilde{d}]_\mu^{(2)}$ $\hat{F}_\mu^{(2)} = -\sqrt{6/5}[p^\dagger\tilde{d} - d^\dagger\tilde{p}]_\mu^{(2)} - \sqrt{14/5}[d^\dagger\tilde{f} - f^\dagger\tilde{d}]_\mu^{(2)}$ $\quad + (\sqrt{7}/10)(\sqrt{21}[p^\dagger\tilde{p}]_\mu^{(2)} - 5[d^\dagger\tilde{d}]_\mu^{(2)} - \sqrt{6}[f^\dagger\tilde{f}]_\mu^{(2)} + 2\sqrt{6}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(2)})$ $\hat{T}_\mu^{(3)} = \sqrt{3/5}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(3)} - (1/\sqrt{2})[d^\dagger\tilde{d}]_\mu^{(3)} + \sqrt{3/10}[f^\dagger\tilde{f}]_\mu^{(3)}$ $\hat{\mathscr{D}}_\mu^{(4)} = \frac{1}{2}([d^\dagger\tilde{d}]_\mu^{(4)} + \sqrt{2/5}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(4)} - \sqrt{11/5}[f^\dagger\tilde{f}]_\mu^{(4)})$ $\hat{C}_{2,SU_{pdf}(6)} = \frac{1}{90}\hat{N}^{(0)} \cdot \hat{N}^{(0)} + \frac{1}{120}\hat{L}^{(1)} \cdot \hat{L}^{(1)} + \frac{1}{3}\hat{\mathscr{D}}^{(2)} \cdot \hat{\mathscr{D}}^{(2)} + \frac{1}{45}\hat{F}^{(2)} \cdot \hat{F}^{(2)}$ $\quad + (1/3\sqrt{7})(\hat{\mathscr{D}}^{(2)} \cdot \hat{F}^{(2)} + \hat{F}^{(2)} \cdot \hat{\mathscr{D}}^{(2)})$ $\quad - \frac{1}{6}\hat{M}^{(2)} \cdot \hat{M}^{(2)} + \frac{10}{57}\hat{T}^{(3)} \cdot \hat{T}^{(3)} + \frac{1}{2}\hat{\mathscr{D}}^{(4)} \cdot \hat{\mathscr{D}}^{(4)}$
$U_d(5)$	$[d^\dagger\tilde{d}]_0^{(0)} = (1/\sqrt{5})\hat{n}_d \qquad [d^\dagger\tilde{d}]_\mu^{(1)} = (1/\sqrt{10})\hat{L}_\mu^{(1)} \qquad [d^\dagger\tilde{d}]_\mu^{(2,3,4)}$ $\hat{C}_{2,SU_d(5)} = \frac{1}{10}\sum_{K=1}^{4}[d^\dagger\tilde{d}]^{(K)} \cdot [d^\dagger\tilde{d}]^{(K)} = \frac{2}{25}\hat{n}_d(\hat{n}_d + 5)$
$U_{pf}(5)$	$\hat{H}_0^{(0)} = (2/\sqrt{5})(\hat{n}_p + \hat{n}_f) = (2/\sqrt{5})\hat{N}_-$ $\hat{H}_\mu^{(1)} = (1/\sqrt{10})(\sqrt{2}[p^\dagger\tilde{p}]_\mu^{(1)} + 2\sqrt{7}[f^\dagger\tilde{f}]_\mu^{(1)}) = (1/\sqrt{10})\hat{L}_{\mu,pf}^{(1)}$ $\hat{H}_\mu^{(2)} = \frac{1}{5}(-2\sqrt{6}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(2)} - \sqrt{21}[p^\dagger\tilde{p}]_\mu^{(2)} + \sqrt{6}[f^\dagger\tilde{f}]_\mu^{(2)})$ $\hat{H}_\mu^{(3)} = -\sqrt{3/5}(\sqrt{2}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(3)} + [f^\dagger\tilde{f}]_\mu^{(3)})$ $\hat{H}_\mu^{(4)} = (1/\sqrt{5})(\sqrt{2}[p^\dagger\tilde{f} + f^\dagger\tilde{p}]_\mu^{(4)} - \sqrt{11}[f^\dagger\tilde{f}]_\mu^{(4)})$ $\hat{C}_{2,SU_{pf}(5)} = \frac{1}{10}\sum_{K=1}^{4}\hat{H}^{(K)} \cdot \hat{H}^{(K)}$
$U_{pdf}(5)$	$\hat{F}_0^{(0)} = (1/\sqrt{5})(2\hat{n}_p + \hat{n}_d + 2\hat{n}_f) \qquad \hat{F}_\mu^{(1)} = (1/\sqrt{10})(\hat{L}_p + \hat{L}_d + \hat{L}_f)$ $\hat{F}_\mu^{(2)} = -2\hat{\mathscr{D}}_\mu^{(2)} = [d^\dagger\tilde{d}]_\mu^{(2)} + \hat{H}_\mu^{(2)} \qquad \hat{F}_\mu^{(3)} = -\sqrt{2}\hat{T}_\mu^{(3)} = [d^\dagger\tilde{d}]_\mu^{(3)} + \hat{H}_\mu^{(3)}$ $\hat{F}_\mu^{(4)} = 2\hat{\mathscr{D}}_\mu^{(4)} = [d^\dagger\tilde{d}]_\mu^{(4)} + \hat{H}_\mu^{(4)}$ $\hat{C}_{2,SU_{pdf}(5)} = \frac{1}{10}\sum_{K=1}^{4}\hat{F}^{(K)} \cdot \hat{F}^{(K)} = \hat{C}_{2,SU_d(5)} + \hat{C}_{2,SU_{pf}(5)} + \frac{1}{5}\sum_{K=1}^{4}[d^\dagger\tilde{d}]^{(K)} \cdot \hat{H}^{(K)}$
$O_{pdf}(5)$	$\hat{F}_\mu^{(1)} = (1/\sqrt{10})(\hat{L}_p + \hat{L}_d + \hat{L}_f) \qquad \hat{F}_\mu^{(3)}$ $\hat{C}_{2,O_{pdf}(5)} = \frac{1}{3}\sum_{K=\mathrm{odd}}\hat{F}^{(K)} \cdot \hat{F}^{(K)} = \frac{1}{3}(\frac{1}{10}\hat{L}_{(pdf}^2 + \hat{F}^{(3)} \cdot \hat{F}^{(3)})$
$O_{pdf}(3)$	$\hat{L}_{pdf,\mu}^{(1)} = \sqrt{10}\hat{F}_\mu^{(1)},\ \hat{C}_2 = 5F^{(1)} \cdot F^{(1)} = \frac{1}{2}\hat{L}_{pdf} \cdot \hat{L}_{pdf}$

implicit, and providing $N_{pdf} - 2n_1 - 3n_2 \geqslant 0$. The quadratic Casimir operator has the expectation value in the representation $[0, \mu, 0, \nu, 0]$

$$\langle\hat{C}_{2,SU_{pdf}(6)}\rangle = \tfrac{1}{9}[\mu^2 + \nu^2 + \mu\nu + 6(\mu + \nu)]. \tag{12}$$

The $U_d(5)$ subalgebra is the usual d-boson algebra for vibrational nuclei, while the $U_{pf}(5)$ subalgebra is the algebra obtained using $j_1 = j_2 = j_3 = j_4 = 2$ in (2) and projecting onto the negative-parity bosons [6]. The branching rules for $U_d(5)$ are discussed in the appendix. In terms of Dynkin labels, the branching rules for $SU_{pf}(10) \supset SU_{pdf}(5)$, for fully symmetric representations of $SU_{pf}(10)$, are $[N_{pf}] \rightarrow [\nu, 0, N_{pf} - 2\nu, 0]$, where

$\nu=0,1,\ldots,[N_{pf}/2]$. The quadratic Casimir operators then have the expectation values

$$\langle \hat{C}_{2,SU_d(5)}\rangle = \tfrac{2}{25}N_d(N_d+5) \tag{13}$$

$$\langle \hat{C}_{2,SU_{pf}(5)}\rangle = \tfrac{1}{25}[3N_{pf}(N_{pf}+5)+10\nu(\nu-N_{pf}-1)]. \tag{14}$$

The $U_{pdf}(5)$ subalgebra is obtained by combining generators of the same rank from the $U_d(5)$ and $U_{pf}(5)$ subalgebras. The relation between the construction of the generators from $U_d(5)\oplus U_{pf}(5)$ and $SU_{pdf}(6)$ are indicated in table 3. The representations of $SU_{pdf}(5)$ are obtained by coupling the Young tableaux of $SU_d(5)$ and $SU_{pf}(5)$ given above. The result in Young labels for $SU_d(5)\oplus SU_{pf}(5)\supset SU_{pdf}(5)$ is $[N_d]\otimes[n_1,n_2,n_2,0]\to[n_1+m_1,n_2+m_2,n_2,m_3]$, where $m_1+m_2+m_3=N_d$, $m_1=\max(0,N_d-n_1),\ldots,N_d$, $m_2=0,\ldots,\min(N_d-m_1,\quad n_1-n_2)$ and $0\leqslant m_3=N_d-m_1-m_2\leqslant n_2$. The branching rules for the decomposition $SU_{pdf}(6)\supset SU_{pdf}(5)$ in Dynkin labels are obtained by the rules $[0,n_1,0,n_2,0]\to[i,n_1-i,n_2-j,j]$, where $i=0,1,\ldots,n_1$ and $j=0,1,\ldots,n_2$. For certain representations of O(10) the branching rules $O(10)\supset SU_{pdf}(5)$ are straightforward to establish. For example, $(0,0,0,0,v_{10})\to[0,v_{10}-n_1-i,0,n_1]$, where $n_1=0,1,\ldots,v_{10}-i$ and $i=0,1,\ldots,v_{10}$, and $(v_{10},0,0,0,0)\to[v_{10}-n_1,0,0,n_1]$, where $n_1=0,1,\ldots,v_{10}$. The quadratic Casimir operator in Young labels $[l_1,l_2,l_3,l_4]$ has the expectation value

$$\langle \hat{C}_{2,SU_{pdf}(5)}\rangle = \tfrac{1}{25}[2(l_1^2+l_2^2+l_3^2+l_4^2)-(l_1l_2+l_2l_3+l_3l_4+l_1l_3+l_1l_4+l_2l_4) \\ +5(2l_1+l_2-l_4)]. \tag{15}$$

The 10 generators of $O_{pdf}(5)$ are identified as the odd rank tensors of $SU_{pdf}(5)$. Methods for computing the branching rules $[n_1,n_2,n_3,n_4]\to(l_1,l_2)$ for $SU_{pdf}(5)\supset O_{pdf}(5)$ can be found in [14]. The quadratic Casimir expectation value in Young labels is

$$\langle \hat{C}_{2,O_{pdf}(5)}\rangle = \tfrac{1}{6}[l_1(l_1+3)+l_2(l_2+1)]. \tag{16}$$

The final subalgebra in this chain is the $O_{pdf}(3)$ subalgebra, which is generated by $\hat{F}^{(1)}=(1/\sqrt{10})\hat{L}_{pdf}$. As before, the Casimir operator has the value $\langle\hat{C}_2\rangle=\frac{1}{2}L(L+1)$. Branching rules $(l_1,l_2)\to L$ for $O_{pdf}(5)\supset O_{pdf}(3)$ can be obtained with the generating functions of [15].

4.2. *The $SU_{spdf}(3)$ limit (IIa)*

This dynamical symmetry is the rotational limit of the model and is shown in table 3 and the generators in table 5. It consists only of class A algebras. As a consequence, there is no natural octupole deformation in this limit, in spite of the fact that SU(3) is conventionally associated with deformation. The $U_{pf}(10)\oplus U_{sd}(6)$ subalgebra was already discussed. The next subalgebras are SU(3), which are realized by the usual Elliott generators [16]. The branching rules $[N_{pf}]\to(\lambda_2,\mu_2)$ for $U_{pf}(10)\supset SU_{pf}(3)$ have been partially tabulated [5], while $[N_d]\to(\lambda_1,\mu_1)$ for $U_{sd}(6)\supset SU_{sd}(3)$ are well known [3]. The usual form of this expectation value is given in terms of the Elliott (or Dynkin) labels (λ,μ)

$$\langle \hat{C}_{2,SU(3)}\rangle = \tfrac{1}{9}[\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)]. \tag{17}$$

The $SU_{sd}(3)\oplus SU_{pf}(3)$ symmetry can be broken by combining the generators of the two algebras into one $SU_{spdf}(3)$ algebra. A phase ambiguity that arises because one can add or subtract the quadrupole generators can result in different physics. Although

Table 5. Generators and quadratic invariants of the $SU_{spdf}(3)$ limit (IIa). Refer to table 3 for subalgebra embeddings.

$SU_{sd}(3)$	$\hat{L}^{(1)}_{\mu}=\sqrt{10}[d^{+}\tilde{d}]^{(1)}_{\mu}$ $\quad \hat{Q}^{(2)}_{\mu,sd}=[s^{+}\tilde{d}+d^{+}s]^{(2)}_{\mu}\pm(\sqrt{7}/2)[d^{+}\tilde{d}]^{(2)}_{\mu}$ $\hat{C}_{2,SU_{sd}(3)}=\frac{1}{9}(2\hat{Q}_{sd}\cdot\hat{Q}_{sd}+\frac{3}{4}\hat{L}_{d}\cdot\hat{L}_{d})$
$SU_{pf}(3)$	$\hat{L}^{(1)}_{\mu}=\sqrt{2}[p^{+}\tilde{p}]^{(1)}_{\mu}+2\sqrt{7}[f^{+}\tilde{f}]^{(1)}_{\mu}$ $\hat{Q}^{(2)}_{\mu,pf}=(3\sqrt{7}/5)[p^{+}\tilde{f}+f^{+}p]^{(2)}_{\mu}\pm\{(9\sqrt{3}/10)[p^{+}\tilde{p}]^{(2)}_{\mu}+(3\sqrt{42}/10)[f^{+}\tilde{f}]^{(2)}_{\mu}\}$ $\hat{C}_{2,SU_{pf}(3)}=\frac{1}{9}(2\hat{Q}_{pf}\cdot\hat{Q}_{pf}+\frac{3}{4}\hat{L}_{pf}\cdot\hat{L}_{pf})$
$SU_{spdf}(3)$	$\hat{L}^{(1)}_{\mu,spdf}=\hat{L}^{(1)}_{\mu,d}+\hat{L}^{(1)}_{\mu,pf}$ $\quad \hat{Q}^{(2)}_{\mu,spdf}=\hat{Q}^{(2)}_{\mu,sd}\pm\hat{Q}^{(2)}_{\mu,pf}$ $\hat{C}_{2,SU_{spdf}(3)}=\frac{1}{9}[2(\hat{Q}_{sd}\pm\hat{Q}_{pf})^{(2)}\cdot(\hat{Q}_{sd}\pm\hat{Q}_{pf})^{(2)}+\frac{3}{4}\hat{L}\cdot\hat{L}]$
$O_{d}(3)$	$\hat{L}^{(1)}_{d,\mu}=\sqrt{10}[d^{+}\tilde{d}]^{(1)}_{\mu}$ $\quad \hat{C}_{2,O(3)}=\frac{1}{2}\hat{L}^{(1)}_{d}\cdot\hat{L}^{(1)}_{d}$
$O_{pf}(3)$	$\hat{L}^{(1)}_{pf,\mu}=\sqrt{2}[p^{+}\tilde{p}]^{(1)}_{\mu}+2\sqrt{7}[f^{+}\tilde{f}]^{(1)}_{\mu}$ $\quad \hat{C}_{2,O(3)}=\frac{1}{2}\hat{L}^{(1)}_{pf}\cdot\hat{L}^{(1)}_{pf}$
$O_{pdf}(3)$	$\hat{L}^{(1)}_{pdf,\mu}=\hat{L}^{(1)}_{d,\mu}+\hat{L}^{(1)}_{pf,\mu}$ $\quad \hat{C}_{2,O(3)}=\frac{1}{2}\hat{L}^{(1)}_{pdf}\cdot\hat{L}^{(1)}_{pdf}$

the energy levels are invariant with respect to the sign, the E2 transitions are not. The branching rule for $(\lambda_1,\mu_1)\otimes(\lambda_2,\mu_2)\rightarrow(\lambda,\mu)$ are easily determined from their corresponding Young tableaux.

We note that the $SU_{sd}(3)\oplus SU_{pf}(3)$ symmetry can be broken a second way to $O_{d}(3)\oplus O_{pf}(3)$, generated by the operators $\hat{L}^{(1)}_{d}$ and $L^{(1)}_{pf}$, respectively. The branching rules for $SU(3)\supset O(3)$ are well known [3] and will not be repeated. Each of these algebras has quadratic Casimir expectation value $\langle\hat{C}_{2,O(3)}\rangle=\frac{1}{2}L(L+1)$. Finally, $O_{pdf}(3)$ generated by $\hat{L}^{(1)}_{pdf}$ is common to both $SU_{pdf}(3)$ and $O_{d}(3)\oplus O_{pf}(3)$, and the branching rules are the usual angular momentum couplings.

4.3. *The $SU_{pdf}(3)$ limit (IIb)*

This limit, originally suggested as a limit that contains parity doublets [17], is shown in tables 3 and 6. It contains two class C subalgebras: $SU_{pdf}(6)$ and $SU_{pdf}(3)$. The $SU_{pdf}(3)$ algebra has the feature that the quadrupole operator is of mixed parity, and an E2 operator is contained in the $SU_{pdf}(6)$ algebra. This limit is a strong candidate for a description of octupole deformed systems. The subalgebras and branching rules

Table 6. Generators and quadratic invariants of the $SU_{pdf}(3)$ limit (IIb). Refer to table 3 for subalgebra embeddings.

$U_{pdf}(6)$	(See table 4 for generators and Casimir invariant)
$SU_{pdf}(3)$	$\hat{L}^{(1)}_{pdf,\mu}=\sqrt{2}[p^{+}\tilde{p}]^{(1)}_{\mu}+\sqrt{10}[d^{+}\tilde{d}]^{(1)}_{\mu}+2\sqrt{7}[f^{+}\tilde{f}]^{(1)}_{\mu}$ $\hat{F}^{(2)}_{\mu}=-\sqrt{6/5}[p^{+}\tilde{d}-d^{+}\tilde{p}]^{(2)}_{\mu}-\sqrt{14/5}[d^{+}\tilde{f}-f^{+}\tilde{d}]^{(2)}_{\mu}$ $\quad+(\sqrt{7}/10)(\sqrt{21}[p^{+}\tilde{p}]^{(2)}_{\mu}-5[d^{+}\tilde{d}]^{(2)}_{\mu}-\sqrt{6}[f^{+}\tilde{f}]^{(2)}_{\mu}+2\sqrt{6}[p^{+}\tilde{f}+f^{+}\tilde{p}]^{(2)}_{\mu})$ $\hat{C}_{2,SU_{pdf}(3)}=\frac{1}{9}(2\hat{F}^{(2)}\cdot\hat{F}^{(2)}+\frac{3}{4}\hat{L}_{pdf}\cdot\hat{L}_{pdf})$
$O_{pdf}(3)$	$\hat{L}^{(1)}_{pdf,\mu}$ $\quad \hat{C}_{2,O(3)}=\frac{1}{2}\hat{L}^{(1)}_{pdf}\cdot\hat{L}^{(1)}_{pdf}$

for $U_{pdf}(15)$ and $SU_{pdf}(6)$ have been detailed above. The quadratic Casimir has the expectation value

$$\langle \hat{C}_{2,SU_{spdf}(3)} \rangle = \tfrac{1}{9}[\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)]. \tag{18}$$

The branching rules $[0, \mu, 0, \nu, 0] \rightarrow (\lambda, \mu)$ for $SU_{pdf}(6) \supset SU_{pdf}(3)$ for low-dimensional representations can be found in [18]. The $O_{pdf}(3)$ subalgebra is generated by $\hat{L}^{(1)}_{pdf}$, with the same branching rules referred to in limit IIa.

4.4. *The $SU_{spdf}(4) \sim O_{spdf}(6)$ limit (IIIa)*

This limit corresponds to the coupling of the $O_{sd}(6)$ 'γ-unstable' limit to the $O_{pf}(6)$ limit of the negative parity bosons, as indicated in tables 3 and 7. The only algebra not of class A is $SU(4) \oplus SU(4)$, which is of class B, which can be used to bring about octupole deformations in a γ-unstable system. The subalgebras $U_{sd}(6) \oplus U_{pf}(10)$ and $SU(4) \oplus SU(4)$ were discussed above. The $U_{pf}(4)$ algebra is the algebra obtained with $j_1 = j_2 = j_3 = j_4 = \frac{3}{2}$ in (2) and projecting onto the pf boson subspace [6]. The $O_{pf}(6)$ algebra is obtained by removing the rank-0 number operator $\hat{E}^{(0)}_0 = \hat{n}_p + \hat{n}_f$ from $U_{pf}(4)$. (Projecting onto positive- rather than negative-parity bosons produces the $O_{sd}(6)$ algebra.) In table 7, the pairing operator is $\hat{P}^{\dagger}_{sd} = d^{\dagger} \cdot d^{\dagger} - s^{\dagger} \cdot s^{\dagger}$. The Dynkin label branching rules $[N_{pf}] \rightarrow [n_1, n_2, n_3]$ for representations of $U_{pf}(10) \supset SU_{pf}(4) \sim O_{pf}(6)$ are given by the rule $[N_{pf}] \rightarrow [2l_1 - 6l_2, N_{pf} - l_1 - 4l_3, 2l_2]$, with the restrictions $l_1 = N_{pf}, N_{pf} - 2, \ldots, 3l_2$ or $3l_2 + 1$, $l_2 = 0, 1, \ldots, [N_{pf}/3]$ and $l_3 = 0, 1, \ldots, (N_{pf} - l_1)/4$. The representations are labelled in terms of the SU(4) labels since Young tableau methods for Kronecker products and subalgebra representations are well known. (The relation between the Young labels of SU(4) $[n_1, n_2, n_3]$ and the usual Cartan SO(6) labels (l_1, l_2, l_3) is $n_1 = l_1 - l_2$, $n_2 = l_1 - l_3$ and $n_3 = l_2 - l_3$.) The branching rules $[N_{sd}] \rightarrow (v_{sd}, 0, 0)$

Table 7. Generators and quadratic invariants of the $SU_{spdf}(4) \sim O_{spdf}(6)$ limit (IIIa). Refer to table 3 for subalgebra embeddings.

$O_{sd}(6)$	$\hat{K}^{(1)}_{\mu} = \sqrt{2}[d^{\dagger}\tilde{d}]^{(1)}_{\mu} = (1/\sqrt{5})\hat{L}^{(1)}_{\mu}$ $\quad \hat{K}^{(2)}_{\mu} = [s^{\dagger}\tilde{d} + d^{\dagger}s]^{(2)}_{\mu}$ $\quad \hat{K}^{(3)}_{\mu} = -\sqrt{2}[d^{\dagger}\tilde{d}]^{(3)}_{\mu}$ $\hat{C}_{2,O_{sd}(6)} = \frac{1}{8}\sum_{K>0} \hat{K}^{(K)} \cdot \hat{K}^{(K)} = \frac{1}{8}[\hat{N}_{sd}(\hat{N}_{sd} + 4) - \hat{P}^{\dagger}_{sd} \cdot \hat{P}_{sd}]$
$O_{pf}(6)$	$\hat{E}^{(1)}_{\mu} = (1/\sqrt{5})(\sqrt{2}[p^{\dagger}\tilde{p}]^{(1)}_{\mu} + 2\sqrt{7}[f^{\dagger}\tilde{f}]^{(1)}_{\mu}) = (1/\sqrt{5})\hat{L}^{(1)}_{\mu}$ $\hat{E}^{(2)}_{\mu} = \frac{1}{5}(\sqrt{21}[p^{\dagger}\tilde{f} + f^{\dagger}\tilde{p}]^{(2)}_{\mu} - 2\sqrt{6}[p^{\dagger}\tilde{p}]^{(2)}_{\mu} + 2\sqrt{21}[f^{\dagger}\tilde{f}]^{(2)}_{\mu})$ $\hat{E}^{(3)}_{\mu} = \sqrt{6/5}(\sqrt{2}[p^{\dagger}\tilde{f} + f^{\dagger}\tilde{p}]^{(3)}_{\mu} + [f^{\dagger}\tilde{f}]^{(3)}_{\mu})$ $\hat{C}_{2,O_{pf}(6)} = \frac{1}{8}(\frac{1}{5}\hat{L}^{(1)}_{pf} \cdot \hat{L}^{(1)}_{pf} + \hat{E}^{(2)} \cdot \hat{E}^{(2)} + \hat{E}^{(3)} \cdot \hat{E}^{(3)})$
$O_{spdf}(6)$	$\hat{B}^{(1)}_{\mu} = 2\hat{\mathscr{H}}^{(1)}_{+,\mu} = (1/\sqrt{5})(\sqrt{2}[p^{\dagger}\tilde{p}]^{(1)}_{\mu} + \sqrt{10}[d^{\dagger}\tilde{d}]^{(1)}_{\mu} + 2\sqrt{7}[f^{\dagger}\tilde{f}]^{(1)}_{\mu}) = (1/\sqrt{5})\hat{L}^{(1)}_{\mu}$ $\hat{B}^{(2)}_{\mu} = 2\hat{\mathscr{H}}^{(2)}_{+,\mu} = [s^{\dagger}\tilde{d} + d^{\dagger}s]^{(2)}_{\mu} + (\sqrt{21}/5)[p^{\dagger}\tilde{f} + f^{\dagger}\tilde{p}]^{(2)}_{\mu} - \frac{2}{5}\sqrt{6}[p^{\dagger}\tilde{p}]^{(2)}_{\mu} + \frac{2}{5}\sqrt{21}[f^{\dagger}\tilde{f}]^{(2)}_{\mu}$ $\hat{B}^{(3)}_{\mu} = 2\hat{\mathscr{H}}^{(3)}_{+,\mu} = 2\hat{T}^{(3)}_{\mu} = -\sqrt{2}[d^{\dagger}\tilde{d}]^{(3)}_{\mu} + 2\sqrt{3/5}[p^{\dagger}\tilde{f} + f^{\dagger}\tilde{p}]^{(3)}_{\mu} + \sqrt{6/5}[f^{\dagger}\tilde{f}]^{(3)}_{\mu}$ $\hat{C}_{2,O_{spdf}(6)} = \frac{1}{8}(\frac{1}{5}\hat{L}^{(1)}_{pdf} \cdot \hat{L}^{(1)}_{pdf} + \hat{B}^{(2)} \cdot \hat{B}^{(2)} + \hat{B}^{(3)} \cdot \hat{B}^{(3)})$
$O_{pdf}(5)$	$\hat{B}^{(1)}_{\mu} = (1/\sqrt{5})(\sqrt{2}[p^{\dagger}\tilde{p}]^{(1)}_{\mu} + \sqrt{10}[d^{\dagger}\tilde{d}]^{(1)}_{\mu} + 2\sqrt{7}[f^{\dagger}\tilde{f}]^{(1)}_{\mu}) = (1/\sqrt{5})\hat{L}^{(1)}_{pdf,\mu}$ $\hat{B}^{(3)}_{\mu} = 2\hat{T}^{(3)}_{\mu} = \sqrt{2/5}(-\sqrt{5}[d^{\dagger}\tilde{d}]^{(3)}_{\mu} + \sqrt{6}[p^{\dagger}\tilde{f} + f^{\dagger}\tilde{p}]^{(3)}_{\mu} + \sqrt{3}[f^{\dagger}\tilde{f}]^{(3)}_{\mu})$ $\hat{C}_{2,O_{pdf}(5)} = \frac{1}{6}(\frac{1}{5}\hat{L}^{(1)}_{pdf} \cdot \hat{L}^{(1)}_{pdf} + \hat{B}^{(3)} \cdot \hat{B}^{(3)})$
$O_{pdf}(3)$	$\hat{L}^{(1)}_{pdf,\mu}$ $\quad \hat{C}_{2,O(3)} = \frac{1}{2}\hat{L}^{(1)}_{pdf} \cdot \hat{L}^{(1)}_{pdf}$

for $SU_{sd}(6) \supset O_{sd}(6)$ are given in the appendix. The quadratic Casimir invariants in the representations $(v_{sd}, 0, 0)$ and $[n_1, n_2, n_3]$ have the expectation values

$$\langle \hat{C}_{2,O_{sd}(6)} \rangle = \tfrac{1}{8} v_{sd}(v_{sd}+4) \tag{19}$$

$$\langle \hat{C}_{2,O_{pf}(6)} \rangle = \tfrac{1}{16}[3(n_1^2+n_2^2+n_3^2) - 2(n_1 n_2 + n_2 n_3 + n_3 n_1) + 4(3n_1 + n_2 - n_3)]. \tag{20}$$

The $O_{spdf}(6)$ subalgebra appears as a subalgebra of $SU(4) \oplus SU(4)$ and $O_{sd}(6) \oplus O_{pf}(6)$. The $O_{spdf}(6)$ subalgebra of $SU(4) \oplus SU(4)$ is obtained by keeping only the even-parity generators of $SU(4) \oplus SU(4)$, defined by $\hat{\mathscr{H}}_{Q,+}^{(K)}$ $(K = 1, 2, 3)$. The same generators can be obtained by combining the generators of $O_{sd}(6)$ and $O_{pf}(6)$. The quadratic Casimir operator and its expectation value in Young labels are

$$\langle \hat{C}_{2,O_{spdf}(6)} \rangle = \tfrac{1}{8}[l_1(l_1+4) + l_2(l_2+2) + l_3^2]. \tag{21}$$

The branching rules for $O_{sd}(6) \oplus O_{pf}(6) \supset O_{spdf}(6)$ and $SU(4) \oplus SU(4) \supset O_{spdf}(6)$ are obtained by the usual Young tableau methods for SU(4).

The odd-rank tensors form the subalgebra $O_{pdf}(5)$. Since we retain SU(4) labels for the algebras we denote O(6), the branching rules for $SU(4) \sim O_{spdf}(6) \supset O_{pdf}(5)$ in Young labels are $[n_1, n_2, n_3] \to (l_1, l_2)$, where $(n_1+n_2-n_3)/2 \geqslant l_1 \geqslant (n_1-n_2+n_3)/2 \geqslant l_2 \geqslant |n_1-n_2-n_3|/2$. The quadratic Casimir operator then has the expectation value

$$\langle \hat{C}_{2,O_{pdf}(5)} \rangle = \tfrac{1}{6}[l_1(l_1+3) + l_2(l_2+1)]. \tag{22}$$

The remaining subalgebra structure is just the usual $O_{pdf}(3)$ subalgebra, generated by $\hat{B}^{(1)} = (1/\sqrt{5})\hat{L}_{pdf}^{(1)}$. The representations of $O_{pdf}(3)$ contained in the $O_{pdf}(5)$ representations can be found with the generating function of [15].

4.5. The $SU_{pdf}(4)$ limit (IIIb)

This is another SU(4) limit (see tables 3 and 8), but unlike limit (IIIa), the generators here do not commute with the number operator for positive- or negative-parity bosons.

Table 8. Generators and quadratic invariants of the $SU_{pdf}(4)$ limit (IIIb). Refer to table 3 for subalgebra embeddings.

$U_{pdf}(6)$	(See table 4 for generators and Casimir invariant)
$O_{pdf}(15)$	$[p^\dagger\tilde{p}]_M^{(1)}$ $[d^\dagger\tilde{d}]_M^{(1,3)}$ $[f^\dagger\tilde{f}]_M^{(1,3,5)}$ $[d^\dagger\tilde{p}+(-1)^L p^\dagger\tilde{d}]_M^{(1,2,3)}$ $[f^\dagger\tilde{p}-(-1)^L p^\dagger\tilde{f}]_M^{(2,3,4)}$ $[f^\dagger\tilde{d}+(-1)^L d^\dagger\tilde{f}]_M^{(1,\ldots,6)}$ $\hat{C}_{2,O_{pdf}(15)} = \frac{1}{26}[\hat{N}_{pdf}(\hat{N}_{pdf}+13) - \hat{P}_{pdf}^\dagger \cdot \hat{P}_{pdf}]$
$SU_{pdf}(4)$	$\hat{L}_\mu^{(1)} = \sqrt{2}[p^\dagger\tilde{p}]_\mu^{(1)} + \sqrt{10}[d^\dagger\tilde{d}]_\mu^{(1)} + 2\sqrt{7}[f^\dagger\tilde{f}]_\mu^{(1)}$ $\hat{M}_\mu^{(2)} = -\sqrt{3/10}[p^\dagger\tilde{d}+d^\dagger\tilde{p}]_\mu^{(2)} + \sqrt{7/10}[d^\dagger\tilde{f}+f^\dagger\tilde{d}]_\mu^{(2)}$ $\hat{T}_\mu^{(3)} = \sqrt{3/5}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]_\mu^{(3)} - (1/\sqrt{2})[d^\dagger\tilde{d}]_\mu^{(3)} + \sqrt{3/10}[f^\dagger\tilde{f}]_\mu^{(3)}$ $\hat{C}_{2,SU_{pdf}(4)} = \frac{1}{2}(\frac{1}{20}\hat{L}_{pdf}^{(1)} \cdot \hat{L}_{pdf}^{(1)} - \hat{M}^{(2)} \cdot \hat{M}^{(2)} + \hat{T}^{(3)} \cdot \hat{T}^{(3)})$
$O_{pdf}(5)$	$\hat{L}_\mu^{(1)} = \sqrt{2}[p^\dagger\tilde{p}]_\mu^{(1)} + \sqrt{10}[d^\dagger\tilde{d}]_\mu^{(1)} + 2\sqrt{7}[f^\dagger\tilde{f}]_\mu^{(1)}$ $\hat{T}_\mu^{(3)} = \sqrt{3/5}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]_\mu^{(3)} - (1/\sqrt{2})[d^\dagger\tilde{d}]_\mu^{(3)} + \sqrt{3/10}[f^\dagger\tilde{f}]_\mu^{(3)}$ $\hat{C}_{2,O_{pdf}(5)} = \frac{1}{6}(\frac{1}{5}\hat{L}_{pdf}^{(1)} \cdot \hat{L}_{pdf}^{(1)} + 4\hat{T}^{(3)} \cdot \hat{T}^{(3)})$
$O_{pdf}(3)$	$\hat{L}_{pdf,\mu}^{(1)}$ $\hat{C}_{2,O(3)} = \frac{1}{2}\hat{L}_{pdf}^{(1)} \cdot \hat{L}_{pdf}^{(1)}$

There are three class B algebras ($O_{pdf}(15)$, $SU_{pdf}(4)$ and $SU(4)\oplus SU(4)$) and one class C algebra ($SU_{pdf}(6)$). This limit is a candidate for Hamiltonians describing octupole deformed systems. The existence of the $SU_{pdf}(4)$ subalgebra is a result of the same structural zero of the 6-J coefficient (8) that allowed the closure of the O(10) subalgebra. The expectation values in the fully symmetric representation of $O_{pdf}(15)$ and the general representation of SU(4) are

$$\begin{aligned}\langle \hat{C}_{2,O_{pdf}(15)}\rangle &= \tfrac{1}{26}v_{pdf}(v_{pdf}+13)\\ \langle \hat{C}_{2,SU_{pdf}(4)}\rangle &= \tfrac{1}{32}[3(n_1^2+n_2^2+n_3^2)-2(n_1n_2+n_2n_3+n_3n_1)+4(3n_1+n_2-n_3)].\end{aligned} \tag{23}$$

The restrictions on the Dynkin labels for $U_{pdf}(15)\supset O_{pdf}(15)$ are $[N_{pdf}]\rightarrow(v_{pdf}, 0, 0, 0, 0, 0, 0, 0)$, where $v_{pdf}=0, 1, \ldots, N_{pdf}$. The $O_{pdf}(5)$ subalgebra of $SU_{pdf}(4)$ is generated by odd-rank tensors $\hat{L}^{(1)}_{pdf}$ and $\hat{T}^{(3)}$, which is identical to the $O_{pdf}(5)$ algebra discussed in limit IIIa.

4.6. *The $O_{spdf}(4)$ limit (IV)*

This limit, shown in tables 3 and 9, is composed of class B subalgebras (except for $O_{pdf}(3)$). Octupole deformed systems will arise naturally from any Hamiltonian constructed in this limit. The maximal subalgebras $O_{spdf}(16)$, $SU(4)\oplus SU(4)$ and O(10) have been detailed in section 3. Their common subalgebra is $Sp(4)\oplus Sp(4)$, whose construction is most easily seen in terms of the odd-rank generators of $SU(4)\oplus SU(4)$. As with the construction of the $SU(4)\oplus SU(4)$ quadratic Casimir operator, only the contribution of good parity (the sum of the quadratic Casimir operators of each Sp(4) algebra) is of interest. The Young label branching rules for $SU(4)\oplus SU(4)\supset Sp(4)\oplus Sp(4)$ are $[n_1, n_2, n_3]\otimes[n_1, n_2, n_3]\rightarrow\langle l_1, l_2\rangle\otimes\langle l_3, l_4\rangle$, where l_1, $l_3=n_1+j-i$, l_2, $l_4=n_2-n_3-j+i$ and $i=0, 1, \ldots, \min(n_3, n_1-n_2)$. For certain representations of O(10), the branching rules to $Sp(4)\oplus Sp(4)$ have a simple form $(N, 0, 0, 0, 0)\rightarrow\langle 0, l_1\rangle\otimes\langle 0, l_2\rangle$, where $l_1=\kappa, \kappa-2, \ldots, 1$ or 0; $l_2=N-\kappa$, $\kappa=N, N-1, \ldots, 0$; $(0, 0, 0, 0, N)\rightarrow\langle l_1, l_2\rangle\otimes\langle l_1, l_2\rangle$, where $l_1=N, N-2, \ldots, 1$ or 0, $l_2=0, 1, \ldots, [l_1/2]$; and for $N\geqslant 1$, $(N, 0, 0, 0, 1)\rightarrow\langle 1, l_1\rangle\otimes\langle 1, l_2\rangle$, where $l_1=N-2i-l_2-k$, $l_2=0, 1, \ldots, N-2i$, $k=0, 1$ and $i=0, 1, \ldots, [N/2]$. The expectation value of this operator in these Young label representations is

$$\langle \hat{C}_{2,Sp(4)\oplus Sp(4)}\rangle = \tfrac{1}{12}[l_1(l_1+4)+l_2(l_2+2)+l_3(l_3+4)+l_4(l_4+2)]. \tag{24}$$

Table 9. Generators and quadratic invariants of the $O_{spdf}(4)$ limit (IV). Refer to table 3 for subalgebra embeddings.

$Sp(4)\oplus Sp(4)$	$\hat{L}^{(1)}_{\mu}=\sqrt{2}[p^{+}\tilde{p}]^{(1)}_{\mu}+\sqrt{10}[d^{+}\tilde{d}]^{(1)}_{\mu}+2\sqrt{7}[f^{+}\tilde{f}]^{(1)}_{\mu}$ $\hat{D}^{(1)}_{\mu}=\sqrt{5}[s^{+}\tilde{p}+p^{+}s]^{(1)}_{\mu}-2\sqrt{2}[p^{+}\tilde{d}+d^{+}\tilde{p}]^{(1)}+\sqrt{7}[d^{+}\tilde{f}+f^{+}\tilde{d}]^{(1)}_{\mu}$ $\hat{T}^{(3)}_{\mu}=\sqrt{3/5}[p^{+}\tilde{f}+f^{+}\tilde{p}]^{(3)}_{\mu}-(1/\sqrt{2})[d^{+}\tilde{d}]^{(3)}_{\mu}+\sqrt{3/10}[f^{+}\tilde{f}]^{(3)}_{\mu}$ $\hat{O}^{(3)}_{\mu}=\frac{1}{2}[s^{+}\tilde{f}+f^{+}\tilde{s}]^{(3)}_{\mu}+\frac{1}{2}\sqrt{3/5}[p^{+}\tilde{d}+d^{+}\tilde{p}]^{(3)}_{\mu}+\sqrt{3/5}[d^{+}\tilde{f}+f^{+}\tilde{d}]^{(3)}_{\mu}$ $\hat{C}_{2,Sp_a(4)\oplus Sp_b(4)}=\hat{C}_{2,Sp_a(4)}+\hat{C}_{2,Sp_b(4)}=\frac{1}{3}(\frac{1}{20}\hat{L}^2+\hat{T}^2-\frac{1}{20}\hat{D}^2-\hat{O}^2)$
$O_{spdf}(4)$	$\hat{L}^{(1)}_{pdf,\mu}=\sqrt{2}[p^{+}\tilde{p}]^{(1)}_{\mu}+\sqrt{10}[d^{+}\tilde{d}]^{(1)}_{\mu}+2\sqrt{7}[f^{+}\tilde{f}]^{(1)}_{\mu}$ $\hat{D}^{(1)}_{\mu}=\sqrt{5}[s^{+}\tilde{p}+p^{+}s]^{(1)}_{\mu}-2\sqrt{2}[p^{+}\tilde{d}+d^{+}\tilde{p}]^{(1)}+\sqrt{7}[d^{+}\tilde{f}+f^{+}\tilde{d}]^{(1)}_{\mu}$ $\hat{C}_{2,O(4)}=\frac{1}{4}(\hat{L}^2_{pdf}-\hat{D}^2)$
$O_{pdf}(3)$	$\hat{L}^{(1)}_{pdf,\mu}$ $\qquad$ $\hat{C}_{2,O(3)}=\frac{1}{2}\hat{L}^{(1)}_{pdf}\cdot\hat{L}^{(1)}_{pdf}$

The O(4) subalgebra is generated by the two rank-one tensors $\hat{L}^{(1)}$ and $\hat{D}^{(1)}$, and has the Casimir operator expectation value

$$\langle \hat{C}_{2,\mathrm{O}(4)} \rangle = \tfrac{1}{4}[l_1(l_1+2)+l_2^2]. \tag{25}$$

The branching rules for $\mathrm{Sp}(4)\oplus\mathrm{Sp}(4)\supset\mathrm{SU}(2)\oplus\mathrm{SU}(2)\sim\mathrm{O}(4)$ can be obtained from the $\mathrm{O}(5)\sim\mathrm{Sp}(4)\supset\mathrm{SU}(2)$ generating function of [15], or from extensive tables [18, 19].

4.7. The $O_{pf}(5)\oplus O_d(5)$ limit (V)

The $\mathrm{O_{pf}}(5)\oplus\mathrm{O_d}(5)$ limit has been included in the pdf classification of dynamical symmetry limits as the algebras in this limit are closely related to most of the previous limits. The structure of this limit is displayed in table 3. The maximal subalgebras of $\mathrm{U_{spdf}}(16)$ have been discussed in section 3. The subalgebras $\mathrm{U_d}(5)\oplus\mathrm{U_{pf}}(10)$ and $\mathrm{O_{pdf}}(15)$ have been discussed in limits I and IIIb, respectively. The remaining two subalgebras at this level in table 3 are $\mathrm{O_{sd}}(6)\oplus\mathrm{O_{pf}}(10)$ and $\mathrm{O_d}(5)\oplus\mathrm{O_{spf}}(11)$, whose general properties and branching rules are discussed in the appendix. Of the remaining subalgebras, only $\mathrm{O_d}(5)\oplus\mathrm{O_{pf}}(10)$ falls into the classification of the appendix. The algebras $\mathrm{U_d}(5)\oplus\mathrm{U_{pf}}(5)$ and $\mathrm{O_{sd}}(6)\oplus\mathrm{O_{pf}}(6)$ are the same algebras discussed in limits I and IIIa. The algebra $\mathrm{O_d}(5)\oplus\mathrm{O_{pf}}(6)$ is the same as in limit IIIa if we replace $\mathrm{O_{sd}}(6)$ by $\mathrm{O_d}(5)$, and the $\mathrm{U_d}(5)\supset\mathrm{O_d}(5)$ embedding falls into the class of the appendix.

All these algebras can be broken to $\mathrm{O_d}(5)\oplus\mathrm{O_{pf}}(5)$. The $\mathrm{O_d}(5)$ and $\mathrm{O_{pf}}(5)$ subalgebras of $\mathrm{U_d}(5)$ and $\mathrm{U_{pf}}(5)$ are obtained from the odd-rank tensors. Similarly, the $\mathrm{O_d}(5)$ and $\mathrm{O_{pf}}(5)$ subalgebras can be constructed from the odd-rank tensors of $\mathrm{O_{sd}}(6)$ and $\mathrm{O_{pf}}(6)$. Explicitly, $\mathrm{O_d}(5)$ is generated by $[d^\dagger\tilde{d}]^{(1)}_\mu$ and $[d^\dagger\tilde{d}]^{(3)}_\mu$, and $\mathrm{O_{pf}}(5)$ by

$$\begin{aligned}\hat{H}^{(1)}_\mu &= \frac{1}{\sqrt{10}}(\sqrt{2}[p^\dagger\tilde{p}]^{(1)}_\mu + 2\sqrt{7}[f^\dagger\tilde{f}]^{(1)}_\mu) = \frac{1}{\sqrt{10}}\hat{L}^{(1)}_{\mathrm{pf},\mu}\\ \hat{H}^{(3)}_\mu &= \sqrt{3/5}(\sqrt{2}[p^\dagger\tilde{f}+f^\dagger\tilde{p}]^{(3)}_\mu - [f^\dagger\tilde{f}]^{(3)}_\mu).\end{aligned} \tag{26}$$

The quadratic Casimir operators are

$$\begin{aligned}\hat{C}_{2,\mathrm{O_d}(5)} &= \tfrac{1}{3}\sum_{K=\mathrm{odd}}[d^\dagger\tilde{d}]^{(K)}\cdot[d^\dagger\tilde{d}]^{(K)} = \tfrac{1}{6}[\hat{n}_d(\hat{n}_d+3) - \hat{P}^\dagger_d\cdot\hat{P}_d]\\ \hat{C}_{2,\mathrm{O_{pf}}(5)} &= \tfrac{1}{3}\sum_{K=\mathrm{odd}}\hat{H}^{(K)}\cdot\hat{H}^{(K)} = \tfrac{1}{3}(\tfrac{1}{10}\hat{L}^2_{pf} + T^2).\end{aligned} \tag{27}$$

Here $\hat{P}^\dagger_d = d^\dagger\cdot d^\dagger$. The branching rules depend on the intermediate subalgebras. For $\mathrm{O_d}(5)$ the rules are simple since it is a subalgebra of $\mathrm{O_{sd}}(6)$ and $\mathrm{U_d}(5)$, for which both cases are well known [3] and fall into the class discussed in the appendix; $\mathrm{O_{pf}}(5)$ is a subalgebra of $\mathrm{O_{pf}}(6)$, $\mathrm{O_{pf}}(10)$ and $\mathrm{U_{pf}}(5)$. The $\mathrm{O}(6)\sim\mathrm{SU}(4)\supset\mathrm{O}(5)$ branching rules were discussed in limit IIIa. The $\mathrm{U_{pf}}(5)\supset\mathrm{O_{pf}}(5)$ and $\mathrm{O_{pf}}(10)\supset\mathrm{O_{pf}}(5)$ branching rules can be found in standard tables [18, 19]. The expectation values of these operators are then (in Young labels)

$$\langle\hat{C}_{2,\mathrm{O_d}(5)}\rangle = \tfrac{1}{6}v_\mathrm{d}(v_\mathrm{d}+3) \qquad \langle\hat{C}_{2,\mathrm{O_{pf}}(5)}\rangle = \tfrac{1}{6}[l_1(l_1+3)+l_2(l_2+1)]. \tag{28}$$

The $\mathrm{O_d}(5)\oplus\mathrm{O_{pf}}(5)$ algebra can be broken to $\mathrm{O_{pdf}}(5)$ by adding the generators. This produces an algebra which is identical to the $\mathrm{O_{pdf}}(5)$ algebra of limits I, IIIa and IIIb. The representations obtained from the Kronecker products of the O(5) representations can be found in [19]. This algebra can also be broken to $\mathrm{O_d}(3)\oplus\mathrm{O_{pf}}(3)$ by keeping the rank-1 operators $\hat{L}^{(1)}_\mathrm{d}$ and $\hat{L}^{(1)}_\mathrm{pf}$. The $\mathrm{O_{pdf}}(3)$ algebra is then obtained from $\hat{L}^{(1)}_\mathrm{pdf}$.

5. Other dynamical symmetry limits

In the previous sections we have studied the structure of most of the subalgebras that appear in the full algebraic lattice of $U_{spdf}(16)$. In order to complete the classification, we briefly discuss the remaining algebras that have not been previously detailed. A property common to all of these subalgebras is that they either couple pf bosons to sd bosons only at the level of SU(2), or they do not contain good boson angular momentum in any subalgebra with rank $r>1$. These algebras, although not of direct physical importance, are included for completeness and shown in table 10.

5.1. Limits including Sp(8) and Sp(6)

The symplectic limits that contain Sp(8) and Sp(6) are not of central interest but are included for completeness. One of the common features of limits involving these algebras is that they do not contain physical boson angular momentum.

The algebra Sp(8) appears only as a subalgebra of SU(8) in the $SU(8)\oplus SU(2)\supset Sp(8)\oplus SU(2)$ limit of $U_{spdf}(16)$. The 36 generators of Sp(8) can be taken as the odd-rank generators $\mathcal{H}^{(K0)}_{Q0}$ ($K=$ odd), $\mathcal{L}^{(10)}_{Q0}$ and *either* $\hat{\mathcal{I}}^{(K0)}_{Q0}$ ($K=2,3$) *or* $\hat{\mathcal{J}}^{(K0)}_{Q0}$ ($K=2,3$), since neither of these appear in the Casimir operator. The Sp(6) subalgebra of Sp(8) is generated by the $\hat{\mathcal{H}}^{(K0)}_{Q0}$ ($K=$ odd), and the SU(2) subalgebra of Sp(6) is generated by $\hat{\mathcal{H}}^{(10)}_{Q0}$. The quadratic Casimir operators and their expectation values in Cartan labels are

$$\langle \hat{C}_{2,\mathrm{Sp}(8)}\rangle = \tfrac{1}{20}[l_1(l_1+8)+l_2(l_2+6)] \tag{29}$$

where the branching rules for $SU(8)\supset Sp(8)$ is $[n_1,\ n_2,\ 0,0,0,0,0]\rightarrow\langle n_1-i,\ n_2-i,\ 0,\ 0\rangle$, where $i=0,1,\ldots,n_2$. For the Sp(6) and SU(2) subalgebras

$$\begin{aligned}\langle \hat{C}_{2,\mathrm{Sp}(6)}\rangle &= \tfrac{1}{16}[m_1(m_1+6)+m_2(m_2+4)]\\ \langle \hat{C}_{2,\mathrm{SU}(2)}\rangle &= \tfrac{1}{2}\tilde{j}(\tilde{j}+1)\end{aligned} \tag{30}$$

with the Young labels for the reduction $Sp(8)\supset Sp(6)\oplus SU(2)$ being given by $\langle l_1, l_2, 0, 0\rangle\rightarrow\langle l_1-j-k,\ l_2-i-k,\ 0\rangle\otimes\tilde{j}$, where $2\tilde{j}=i+j$, $i=0,1,\ldots,l_2$, $j=0,1,\ldots,l_1-l_2$ and $k=0,1,\ldots,n_2-i$.

The $SU(8)\oplus SU(2)$ symmetry can be broken a second way, via $SU(8)\oplus SU(2)\supset SU(6)\oplus SU(2)\oplus SU(2)$. The generators $\hat{\mathcal{H}}^{(K0)}_{Q0}$ and $\hat{\mathcal{L}}^{(10)}_{Q0}$ form the $SU(6)\oplus SU(2)$ subalgebra of SU(8). (The SU(2) algebra is the same as that in the previous paragraph.) The $SU(6)\supset Sp(6)$ subalgebra is generated by the odd-rank generators $\hat{\mathcal{H}}^{(K0)}_{Q0}$ ($K=$ odd). At this point the decomposition is identical to the discussion of Sp(6) in the previous paragraph, and

$$\langle \hat{C}_{2,\mathrm{SU}(6)}\rangle = \tfrac{1}{16}[m_1(m_1+6)+m_2(m_2+4)] \tag{31}$$

where the Cartan labels for $SU(8)\supset SU(6)\oplus SU(2)$ are $[n_1, n_2, 0, 0, 0, 0, 0]\rightarrow[n_1-j-k,\ n_2-i-j, 0, 0, 0]\otimes\tilde{j}$, with $2\tilde{j}=i+k$, $j=0,1,\ldots,n_2$, $i=0,1,\ldots,n_2-j$ and $k=0,1,\ldots,n_1-n_2$. For the $SU(6)\supset Sp(6)$ decomposition, the Young label branching rules are $[n_1,\ n_2,\ 0,\ 0,\ 0]\rightarrow\langle n_1-i,\ n_2-i,\ 0\rangle$, where $i=0,1,\ldots,n_2$.

A similar decomposition arises for $U_{df}(12)$ in the chain $U_{df}(12)\supset SU(6)\oplus SU(2)\supset Sp(6)\oplus SU(2)\oplus SU(2)$. The $SU(6)\oplus SU(2)$ subalgebra is generated by $\hat{\mathcal{H}}^{(K0)}_{Q0}$ ($K=$ odd) for SU(6) and $\hat{\mathcal{H}}^{(01)}_{0Q}$ for SU(2). Then the SU(6) symmetry can be broken to Sp(6) in the same way discussed in the previous paragraph. The $O_{df}(12)$ algebra also has $Sp(6)\oplus SU(2)$ as a maximal subalgebra.

Table 10. Generators and quadratic invariants of the $SU_{pdf}(15) \supset SU(5) \oplus SU(3)$, $U_{spd}(9) \supset SU(3) \oplus SU(3)$ and $U_{pd}(8) \supset SU(4) \oplus SU(2)$ chains of $U_{spdf}(16)$. Refer to Ref. 8 for subalgebra embeddings.

Sp(8)	$\hat{\mathscr{H}}^{(K0)}_{Q0}(K=\text{odd})$, $\hat{\mathscr{L}}^{(10)}_{Q0}$ and *either* $\hat{\mathscr{I}}^{(K0)}_{Q0}(K=2,3)$ *or* $\hat{\mathscr{J}}^{(K0)}_{Q0}(K=2,3)$
SU(6)⊕SU(2)	$\hat{\mathscr{H}}^{(K0)}_{Q0} \qquad \hat{\mathscr{L}}^{(10)}_{Q0}$
Sp(6)	$\hat{\mathscr{H}}^{(K0)}_{Q0}(K=\text{odd})$
SU(2)	$\hat{\mathscr{H}}^{(10)}_{Q0}$
	$\hat{C}_{2,Sp(8)} = \frac{1}{5}(\hat{\mathscr{L}}^{(10)} \cdot \hat{\mathscr{L}}^{(10)} + \sum_{K=\text{odd}} \hat{\mathscr{H}}^{(K0)} \cdot \hat{\mathscr{H}}^{(K0)}) \qquad \hat{C}_{2,Sp(6)} = \frac{1}{4} \sum_{K=\text{odd}} \hat{\mathscr{H}}^{(K0)} \cdot \hat{\mathscr{H}}^{(K0)}$
	$\hat{C}_{2,SU(6)} = \frac{1}{4} \sum_{K=0}^{5} \hat{\mathscr{H}}^{(K0)} \cdot \hat{\mathscr{H}}^{(K0)} \qquad \hat{C}_{2,SU(2)} = \frac{1}{2} \hat{\mathscr{L}}^{(10)} \cdot \hat{\mathscr{L}}^{(10)}$
$U_{spd}(9) \supset Su(3) \oplus SU(3)$	
	$\hat{\mathscr{Q}}^{(0)}_{0,+} = \frac{2}{3}(\hat{n}_p + \hat{n}_d + \hat{n}_f) \qquad \hat{\mathscr{Q}}^{(1)}_{Q,+} = (1/\sqrt{6})(\hat{L}^{(1)}_p + \hat{L}^{(1)}_d)$
SU(3)⊕SU(3)	$\hat{\mathscr{Q}}^{(1)}_{Q,-} = -\frac{2}{3}[s^\dagger \tilde{p} + p^\dagger s]^{(1)}_Q + (\sqrt{5}/3)[p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(1)}_Q$
	$\hat{\mathscr{Q}}^{(2)}_{Q,+} = -(1/\sqrt{3})[p^\dagger \tilde{p}]^{(1)}_Q + (\sqrt{7}/3)[d^\dagger \tilde{d}]^{(2)}_Q + \frac{2}{3}[s^\dagger \tilde{d} + d^\dagger s]^{(2)}_Q$
	$\hat{\mathscr{Q}}^{(2)}_{Q,-} = [p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(2)}$
	$C_{2,SU(3)\oplus SU(3)} = \frac{1}{4} \sum_{K=1,2} (\hat{\mathscr{Q}}^{(K)}_{+} \cdot \hat{\mathscr{Q}}^{(K)}_{+} - \hat{\mathscr{Q}}^{(K)}_{-} \cdot \hat{\mathscr{Q}}^{(K)}_{-})$
$SU_{spd}(3)$	$\hat{L}^{(1)}_{pd,\mu} = \sqrt{6}\hat{\mathscr{Q}}^{(1)}_{\mu,+}$
	$\hat{\mathscr{Q}}^{(2)}_{\mu} = \frac{3}{2}\hat{\mathscr{Q}}^{(2)}_{\mu,+} = [s^\dagger \tilde{d} + d^\dagger s]^{(2)}_\mu + (\sqrt{7}/2)[d^\dagger \tilde{d}]^{(2)}_\mu - (\sqrt{3}/2)[p^\dagger \tilde{p}]^{(2)}_\mu$
SU(3)	$\hat{\mathscr{Q}}^{(1)}_{Q,+} = (1/\sqrt{6})(\hat{L}^{(1)}_p + \hat{L}^{(1)}_d) \qquad \hat{\mathscr{Q}}^{(2)}_{Q,-} = [p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(2)}$
$O_{spd}(4)$	$\hat{\mathscr{Q}}^{(1)}_{Q,+} = (1/\sqrt{6})(\hat{L}^{(1)}_p + \hat{L}^{(1)}_d)$
	$\hat{\mathscr{Q}}^{(1)}_{Q,-} = -\frac{2}{3}[s^\dagger \tilde{p} + p^\dagger s]^{(1)}_Q + (\sqrt{5}/3)[p^\dagger \tilde{d} + d^\dagger \tilde{p}]^{(1)}_Q$
$U_{pdf}(15) \supset SU(5) \oplus SU(3)$	
	$\hat{\mathscr{M}}^{(10)}_{Q0} = \frac{1}{2}\sqrt{3/10}\, \hat{L}^{(1)}_{p,Q} + \frac{1}{6}\sqrt{5/6}\hat{L}^{(1)}_{d,Q} + \frac{1}{3}\sqrt{2/15}\hat{L}^{(1)}_{f,Q}$
	$\quad -(1/2\sqrt{5})[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(1)}_Q - \frac{1}{3}\sqrt{7/15}[d^\dagger \tilde{f} - f^\dagger \tilde{d}]^{(1)}_Q$
	$\hat{\mathscr{M}}^{(20)}_{Q0} = (\sqrt{7}/10)[p^\dagger \tilde{p}]^{(2)}_Q + (1/2\sqrt{3})[d^\dagger \tilde{d}]^{(2)}_Q + (2\sqrt{2}/5)[f^\dagger \tilde{f}]^{2}_Q + (1/5\sqrt{3})[p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(2)}_Q$
	$\quad -\frac{1}{2}\sqrt{7/15}[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(1)}_Q - \sqrt{2/15}[d^\dagger \tilde{f} - f^\dagger \tilde{d}]^{(2)}_Q$
SU(5)	$\hat{\mathscr{M}}^{(30)}_{Q0} = (1/\sqrt{5})([f^\dagger \tilde{f}]^{(3)}_Q + (1/\sqrt{3})[p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(3)}_Q$
	$\quad -[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(3)}_Q - \sqrt{2/3}[d^\dagger \tilde{f} - f^\dagger \tilde{d}]^{(3)}_Q)$
	$\hat{\mathscr{M}}^{(40)}_{Q0} = -(2/3\sqrt{3})[d^\dagger \tilde{d}]^{(4)}_Q + \frac{1}{3}\sqrt{11/15}[f^\dagger \tilde{f}]^{(4)}_Q$
	$\quad +(1/\sqrt{5})[p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(4)}_Q - \frac{1}{3}\sqrt{5/3}[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(4)}_Q$
	$\hat{C}_{2,SU(5)} = \frac{3}{10} \sum_{K=1}^{4} \hat{\mathscr{M}}^{(K0)} \cdot \hat{\mathscr{M}}^{(K0)} \qquad \hat{C}_{2,O(5)} = \sum_{K=1,3} \hat{\mathscr{M}}^{(K0)} \cdot \hat{\mathscr{M}}^{(K0)}$
	$\hat{C}_{2,O(3)} = 15\hat{\mathscr{M}}^{(10)} \cdot \hat{\mathscr{M}}^{(10)}$
SU(3)	$\hat{\mathscr{M}}^{(01)}_{0Q} = -(1/2\sqrt{10})\hat{L}^{(1)}_{p,Q} + (1/6\sqrt{10})\hat{L}^{(1)}_{d,Q} + (1/3\sqrt{10})\hat{L}^{(1)}_{f,Q}$
	$\quad +\frac{1}{2}\sqrt{\frac{3}{5}}[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(1)}_Q + \frac{1}{3}\sqrt{7/5}[d^\dagger \tilde{f} - f^\dagger \tilde{d}]^{(1)}_Q$
	$\hat{\mathscr{M}}^{(02)}_{0Q} = (1/10\sqrt{5})[p^\dagger \tilde{p}]^{(2)}_Q - \frac{1}{2}\sqrt{7/15}[d^\dagger \tilde{d}]^{(2)}_Q + \frac{1}{5}\sqrt{7/10}[f^\dagger \tilde{f}]^{(2)}_Q + \frac{1}{5}\sqrt{21/5}[p^\dagger \tilde{f} + f^\dagger \tilde{p}]^{(2)}_Q$
	$\quad -(\sqrt{3}/10)[p^\dagger \tilde{d} - d^\dagger \tilde{p}]^{(1)}_Q + \frac{1}{5}\sqrt{14/3}[d^\dagger \tilde{f} - f^\dagger \tilde{d}]^{(2)}_Q$
	$\hat{C}_{2,SU(3)} = \frac{5}{6} \sum_{K=1,2} \hat{\mathscr{M}}^{(0K)} \cdot \hat{\mathscr{M}}^{(0K)} \qquad \hat{C}_{2,SU(2)} = 5\hat{\mathscr{M}}^{(01)} \cdot \hat{\mathscr{M}}^{(01)}$

Table 10. (continued)

$\mathrm{U_{pd}(8)\supset SU(4)\oplus SU(2)}$	
	$\hat{\mathscr{R}}^{(10)}_{Q0}=(1/4\sqrt{10})(5\hat{L}^{(1)}_{\mathrm{p}}+3\hat{L}^{(1)}_{\mathrm{d}})-\frac{1}{4}[p^{\dagger}\tilde{d}-d^{\dagger}\tilde{p}]^{(1)}_{Q}$
SU(4)	$\hat{\mathscr{R}}^{(20)}_{Q0}=(\sqrt{3}/4)[p^{\dagger}\tilde{p}]^{(2)}_{Q}+(\sqrt{7}/4)[d^{\dagger}\tilde{d}]^{(2)}_{Q}-(\sqrt{3}/4)[p^{\dagger}\tilde{d}-d^{\dagger}\tilde{p}]^{(2)}_{Q}$
	$\hat{\mathscr{R}}^{(30)}_{Q0}=\frac{1}{2}[d^{\dagger}\tilde{d}]^{(3)}_{Q}-\frac{1}{2}\sqrt{3/2}[p^{\dagger}\tilde{d}-d^{\dagger}\tilde{p}]^{(3)}_{Q}$
SU(2)	$\hat{\mathscr{R}}^{(01)}_{0Q}=(1/4\sqrt{2})(\hat{L}^{(1)}_{\mathrm{d}}-\hat{L}^{(1)}_{\mathrm{p}})+(\sqrt{5}/4)[p^{\dagger}\tilde{d}-d^{\dagger}\tilde{p}]^{(1)}_{Q}$
	$\hat{C}_{2,\mathrm{SU}(4)}=\frac{1}{4}\sum_{K=1}^{3}\hat{\mathscr{R}}^{(K0)}\cdot\hat{\mathscr{R}}^{(K0)}$ $\quad\hat{C}_{2,\mathrm{Sp}(4)}=\frac{1}{3}\sum_{K=1,3}\hat{\mathscr{R}}^{(K0)}\cdot\hat{\mathscr{R}}^{(K0)}$,
	$\hat{C}_{2,\mathrm{SU}(2)}=5\hat{\mathscr{R}}^{(10)}\cdot\hat{\mathscr{R}}^{(10)}$ $\quad\hat{C}_{2,\mathrm{SU}(2)}=\hat{\mathscr{R}}^{(01)}\cdot\hat{\mathscr{R}}^{(01)}$

5.2. The $SU_{pdf}(15)\supset SU(3)\oplus SU(5)$ decomposition

It is convenient to realize these generators in terms of the pseudo-spin representation in (2). Denoting by $\hat{\mathscr{M}}^{(K_1K_2)}_{Q_1Q_2}$ the generators obtained by substituting $j_1=j_3=2$ and $j_2=j_4=1$ in (2), the separation into SU(5) and SU(3) generators is natural. (SU(5) is generated by the 24 generators with only left indices $\mathscr{M}^{(K0)}_{Q0}$.) The number operator $\hat{\mathscr{M}}^{(00)}_{00}=(1/\sqrt{15})\hat{N}_{\mathrm{pdf}}$ can be added to either algebra. The odd-rank tensors form the O(5)⊕SU(2) subalgebra of SU(5)⊕SU(3), which is also a maximal subalgebra of $\mathrm{O_{pdf}(15)}$. The Young label branching rules for $\mathrm{SU_{pdf}(15)\supset SU(5)\oplus SU(3)}$ are $[N_{\mathrm{pdf}}]\to[n_1,n_2,n_3]\otimes[n_1-n_3,n_2-n_3]$, where $n_1\geqslant n_2\geqslant n_3\geqslant 0$ and $n_1+n_2+n_3=N_{\mathrm{pdf}}$. The branching rules for SU(5)⊃O(5), O(5)⊃O(3) and SU(3)⊃O(3) were discussed in limits I and II. The expectation values of the quadratic Casimir operators in Young labels are

$$\begin{aligned}\langle\hat{C}_{2,\mathrm{SU}(5)}\rangle&=\tfrac{1}{25}[2(n_1^2+n_2^2+n_3^2)-(n_1n_2+n_1n_3+n_2n_3)+5(2n_1+n_2)]\\ \langle\hat{C}_{2,\mathrm{O}(5)}\rangle&=\tfrac{1}{6}[l_1(l_1+3)+l_2(l_2+1)]\qquad\langle\hat{C}_{2,\mathrm{O}(3)}\rangle=\tfrac{1}{2}L(L+1)\end{aligned}\tag{32}$$

and switching to Elliott (Dynkin) labels $\lambda=n_1-n_2$ and $\mu=n_2-n_3$ for SU(3):

$$\begin{aligned}\langle\hat{C}_{2,\mathrm{SU}(3)}\rangle&=\tfrac{1}{9}[\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)]\\ \langle\hat{C}_{2,\mathrm{SU}(2)}\rangle&=\tfrac{1}{2}J(J+1).\end{aligned}\tag{33}$$

5.3. The $SU_{spd}(3)$, $SU_{pd}(3)$ and $O_{spd}(4)$ subalgebras of $U_{spd}(9)$

The $\mathrm{U_{spd}(9)}$ algebra alone is an interesting algebra since it couples dipole degrees of freedom to collective quadrupole excitations. Analogous to the two distinct SU(4) subalgebras of SU(4)⊕SU(4), the SU(3)⊕SU(3) limit of $\mathrm{U_{spd}(9)}$ admits two distinct SU(3) embeddings, along with an $\mathrm{O_{spd}(4)}$ subalgebra. The most natural representation of $\mathrm{U_{spd}(9)}$ is provided by (2) with $j_1=j_2=j_3=j_4=1$, and with the additional phase $(-1)^{(1/2)(L_1^2+L_2^2)+L_2}$. These generators are denoted $\hat{Q}^{(KK')}_{QQ'}$. Analogous to the SU(4)⊕SU(4) algebra, generators of good parity can be defined by $\hat{\mathscr{Q}}^{(K)}_{Q,+}=\hat{\mathscr{Q}}^{(K0)}_{Q0}+\hat{\mathscr{Q}}^{(0K)}_{0Q}$ and $\hat{\mathscr{Q}}^{(K)}_{Q,-}=-\mathrm{i}(\hat{\mathscr{Q}}^{(K0)}_{Q0}-\hat{\mathscr{Q}}^{(0K)}_{0Q})$. The even-parity quadratic Casimir operator is constructed from the sum of the two quadratic Casimir operators of each SU(3). The Young branching rules for the decomposition of $\mathrm{U_{spd}(9)}$ into SU(3)⊕SU(3) are given by $[N_{\mathrm{tot}}]\supset[n_1,n_2]\otimes[n_1,n_2]$, where $n_1+n_2=N_{\mathrm{tot}}-3\kappa$, $n_1\geqslant n_2\geqslant 0$ and $\kappa=0,1,\ldots,$

$[N_{tot}/3]$. The expectation value is

$$\langle \hat{C}_{2,SU(3)\oplus SU(3)} \rangle = \tfrac{2}{9}[\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu)] \tag{34}$$

where $\lambda = n_1 - n_2$ and $\mu = n_2$.

There are three subalgebras of SU(3)$\oplus$SU(3), which we call $SU_{spd}(3)$, $SU_{pd}(3)$ and $O_{spd}(4)$. $SU_{spd}(3)$ and $SU_{pd}(3)$ correspond to the embedding of the (02)$\oplus$(01) and (11)$\oplus$(00) representations into the (01)$\otimes$(01) representation of SU(3)$\oplus$SU(3). $SU_{spd}(3)$ is formed by the quadrupole and angular momentum operators: $\sqrt{6}\hat{\mathscr{Q}}^{(1)}_{\mu,+}$ and $(\tfrac{3}{2})\hat{\mathscr{Q}}^{(2)}_{\mu,+}$. This SU(3) algebra corresponds to the coupling of the two SU(3) algebras. The only subalgebra is generated by $\hat{L}^{(1)}_{pd}$.

The $O_{spd}(4)$ subalgebra is generated by the angular momentum operator $\hat{\mathscr{Q}}^{(1)}_{+}$ and the dipole operator $\hat{\mathscr{Q}}^{(1)}_{-}$. Branching rules correspond to the usual SU(3) $\supset$ SU(2) rules mentioned in limit II. The subalgebra of $O_{spd}(4)$ is generated by $\hat{\mathscr{Q}}^{(1)}_{+}$.

The remaining subalgebra of SU(3)$\oplus$SU(3) is $SU_{pd}(3)$, and is generated by the angular momentum operator $\hat{\mathscr{Q}}^{(1)}_{+}$ and the odd-parity quadrupole operator $\hat{\mathscr{Q}}^{(2)}_{-}$. This algebra has the same Casimir operator expectation value (17) discussed in limit II. This algebra closes on SU(3) as a result of the structural zero of the $6-J$ coefficient

$$\begin{Bmatrix} 2 & 2 & 3 \\ 2 & 2 & 1 \end{Bmatrix} = 0. \tag{35}$$

Further, this $SU_{pd}(3)$ subalgebra is also a maximal subalgebra of $O_{pd}(8)$.

5.4. *The* $U_{pd}(8) \supset U(4) \oplus U(2)$ *subalgebra*

The $U_{pd}(8)$ algebra can be represented by the generators of (2) with $j_1 = j_3 = \tfrac{3}{2}$ and $j_2 = j_4 = \tfrac{1}{2}$, which we refer to as $\hat{\mathscr{R}}^{(K_1K_2)}_{Q_1Q_2}$. As usual, the left and right indices generate SU(4) and SU(2). The rank-0 operator $\hat{\mathscr{R}}^{(00)}_{00} = (1/2\sqrt{2})\hat{N}_{pd}$ can be added to either algebra. The subalgebra structure of SU(4) is SU(4) $\supset$ Sp(4) $\supset$ SU(2), where Sp(4) is generated by the odd-rank tensors of SU(4). Branching rules to Sp(4) and SU(2) are identical to those in limit IV. The Cartan label branching rules for the reduction $SU_{pd}(8) \supset SU(4) \oplus SU(2)$ is $[N_{pd}] \to [(N_{pd}/2)+\tilde{j}, (N_{pd}/2)-\tilde{j}, 0] \otimes [\tilde{j}]$, where $\tilde{j} = N_{pd}/2, (N_{pd}/2)-1, \ldots, \tfrac{1}{2}$ or 0. The expectation values of the quadratic Casimir operators are

$$\begin{aligned} &\langle \hat{C}_{2,SU(4)} \rangle = \tfrac{1}{32}[3(n_1^2+n_2^2)-2n_1n_2+4(3n_1+n_2)] \\ &\langle \hat{C}_{2,Sp(4)} \rangle = \tfrac{1}{12}[l_1(l_1+4)+l_2(l_2+2)] \qquad \langle \hat{C}_{2,SU(2)} \rangle = \tfrac{1}{2}L(L+1). \end{aligned} \tag{36}$$

5.5. $O(7) \supset G_2 \supset O(3)$

The only exceptional subalgebra that appears in the $U_{spdf}(16)$ lattice is the usual G_2 subalgebra of $O_f(7)$, the f-boson algebra. The decomposition is $U_f(7) \supset O_f(7) \supset G_2 \supset O_f(3)$, and has been studied in detail [20]. $O_f(7)$ is generated by the odd-rank tensors $[f^\dagger \tilde{f}]^{(K)}$ $(K = 1, 3, 5)$ and G_2 is generated by the tensors with $K = 1, 5$. The quadratic Casimir operator of G_2 is equal to $\tfrac{2}{3}$ times the quadratic Casimir operator of $O_f(7)$, hence G_2 has no separate dynamics. Since the representations of $O_f(7)$ are fully symmetric $(v, 0, 0)$, the branching rules for $O_f(7) \supset G_2$ are $(v, 0, 0) \to (0, v)$.

6. Conclusions

We have explicitly constructed all the dynamical symmetry limits of $U_{spdf}(16)$. The complex structure of $U_{spdf}(16)$ can be seen to reduce to seven pdf dynamical symmetry limits, which we studied in some detail. By introducing three classes of algebras, the pdf algebras can be classified in terms of which algebras provide descriptions of octupole deformation and so forth. The explicit construction of the generators provides the structure of transition operators in the dynamical symmetry limits. The framework is now available to construct dynamical symmetry Hamiltonians from the Casimir invariants. Although many branching rules have been established, the full state-labelling problem for the subalgebras of $U_{spdf}(16)$ has not been completely solved. In practice one can resort to existing computational schemes to obtain branching ratios. However, with the explicit constructions of the Casimir operators and generators, existing numerical methods [21] can be employed to solve the general dynamical symmetry and broken symmetry U(16) Hamiltonian. These calculations, together with electromagnetic transition strengths, are further steps that are necessary to establish the physical relevance of the dynamical symmetry limits.

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Appendix

More than one third of the subalgebras of $U_{spdf}(16)$ are of two basic forms. Specifically, we refer to the subalgebras of the form $U(\Sigma\,(2l_i+1))$ and their subalgebras of the form $O(\Sigma\,(2l_i+1))$. The representations are always fully symmetric, leading to simple expressions for general branching rules and Casimir operator expectation values. Since there are 14 groupings of s, p, d and f bosons (p, d, f, sp, sd, sf, pd, pf, df, spd, spf, sdf, pdf, spdf), there are a total of 28 unitary and orthogonal subalgebras that fall into this class. These correspond to $U(n)$ and $O(n)$ with $n = 3, \ldots, 13$, 15, 16 and with two $n = 8$ values.

A.1. Unitary algebras

Consider the set of boson creation and annihilation operators with spin $\{l_1, l_2, \ldots\}$. The n^2 $(n = \Sigma_{l_i}\,(2l_i+1))$ generators of $U(n)$ are $[b^\dagger_{l_i}\tilde{b}_{l_j}]^{(L)}_M$ with $L = |l_i - l_j|, \ldots, l_i + l_j$ (all l_i, l_j). The quadratic Casimir operator for $SU(n)$ is

$$C_{2,SU(n)} = \frac{n-1}{2n^2}\,\hat{N}(\hat{N}+n) \tag{A.1}$$

where $\hat{N} = \Sigma_{l_i}\,\hat{n}_{l_i} = \Sigma_{l_i}\,\sqrt{2l_i+1}[b^\dagger_{l_i}\tilde{b}_{l_i}]^{(0)}_0$. The branching rules are especially simple. The two cases that appear in the $U_{spdf}(16)$ subalgebra structure are $U(n_1) \supset U(n_1-1)$ for which $[N_1, 0, \ldots, 0] \rightarrow [N_2, 0, \ldots, 0]$, with $N_2 = 0, 1, \ldots, N_1$, which refers to the decoupling of an s boson, and $U(n_1+n_2) \supset U(n_1) \oplus U(n_2)$ for which $[N_1, 0, \ldots, 0] \rightarrow [N_2, 0, \ldots, 0] \otimes [N_1 - N_2, 0, \ldots, 0]$ with $N_2 = 0, 1, \ldots, N_1$.

A.2. Orthogonal algebras

For every algebra U(n) constructed from the n^2 bilinear products $[b^\dagger_{l_1}\tilde{b}_{l_2}]^{(L)}_M$, the O($n$) subalgebra is given by the $\frac{1}{2}n(n-1)$ generators $[b^\dagger_{l_1}\tilde{b}_{l_1}]^{(L)}_M$ ($L=$odd) and $[b^\dagger_{l_1}\tilde{b}_{l_2}+(-1)^{P(L,l_1,l_2)}b^\dagger_{l_2}\tilde{b}_{l_1}]^{(L)}_M$, where $L=l_1-l_2,\ldots,l_1+l_2$ for all l_1 and l_2 with $l_1\geqslant l_2$. Here the phase P must satisfy the relation

$$P(L_3,l_1,l_3)=P(L_1,l_1,l_2)+P(L_2,l_2,l_3)+L_1+L_2+L_3+1 \tag{A.2}$$

imposed by requiring the O(n) algebra to close. A convenient choice of P, which retains the form of the $O_{sd}(6)$ algebra of the IBM-1 model and the $O_{sp}(4)$ algebra of the Vibron model [22], is provided by

$$P(L,l_1,l_2)=\begin{cases}0 & \text{if } l_1 \text{ or } l_2=0\\ L+l_1+l_2+1 & \text{if } l_1 \text{ and } l_2>0.\end{cases} \tag{A.3}$$

The quadratic Casimir operator of O(n) where $n=\Sigma_{l_i}(2l_i+1)$ is

$$\hat{C}_{2,O(n)}=\frac{1}{2(n-2)}\left(\hat{N}(\hat{N}+n-2)-\sum_l \hat{P}^\dagger_l\cdot\hat{P}_l+\sum_{\substack{l,l',K\\ l\neq l'}}(-1)^{l+l'+P(K,l,l')}[b^\dagger_l\tilde{b}_{l'}]^{(K)}\cdot[b^\dagger_{l'}\tilde{b}_l]^{(K)}\right)$$

$$=\frac{1}{2(n-2)}[\hat{N}(\hat{N}+n-2)-\hat{P}^\dagger\cdot\hat{P}] \tag{A.4}$$

where we use the usual definition of the pairing operators for bosons of spin l: $\hat{P}^\dagger_l=b^\dagger_l\cdot b^\dagger_l$ and $\hat{P}_l=\tilde{b}_l\cdot\tilde{b}_l$. The choice of phase does not effect the expectation value of C_2, but only the form of the pairing operator P. For example, the pairing operator for O(16) using $P(K,l_1,l_2)=K+1$ is $P^\dagger=s^\dagger\cdot s^\dagger+p^\dagger\cdot p^\dagger+d^\dagger\cdot d^\dagger+f^\dagger\cdot f^\dagger$, while for $P(K,l_1,l_2)$ as in (A.3), $\hat{P}^\dagger=-s^\dagger\cdot s^\dagger-p^\dagger\cdot p^\dagger+d^\dagger\cdot d^\dagger-f^\dagger\cdot f^\dagger$. Since the representations are fully symmetric, the expectation value of the quadratic Casimir operator for O(n) in the Cartan–Weyl basis is $v(v+n-2)/2(n-2)$, and we have

$$\langle\hat{P}^\dagger\cdot\hat{P}\rangle=N(N+n-2)-v(v+n-2). \tag{A.5}$$

Equation (A.5) illustrates the relation between the pairing operator $\hat{P}^\dagger$ and boson seniority, given by $(N-v)/2$. The branching rules for $U(n)\supset O(n)$ given by $[N,0,\ldots,0]\to(v,0,\ldots,0)$ with $v=N,N-2,\ldots,1$ or 0, and for $O(n_1+n_2)\supset O(n_1)\oplus O(n_2)$ given by $(v_1,0,\ldots,0)\to(v_1-2k-v_2,0,\ldots,0)\otimes(v_2,0,\ldots,0)$ where $v_2=0,1,\ldots,v_1-2k$ and $k=0,1,\ldots,[v_1/2]$.

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